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Kähler Manifolds and the Calabi-Yau Theorem

Bachelor's thesis by

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Abstract

Kähler and in particular Calabi-Yau manifolds have received a great amount of attention for their rich structure and applications in algebraic geometry and mathematical physics. This thesis aims to give an introduction to these topics from the point of view of G-structures by considering the interplay of the various layers of structure on these manifolds as well as their intrinsic torsion and integrability. We also apply these results by giving an overview of the celebrated Calabi-Yau Theorem, its implications and an outline of its proof.

Contents

1 Introduction

The study of Kähler manifolds stands out due to the rich interplay of their various constituent structures. Specifically, they consist of Riemannian, complex and symplectic structures on their tensor algebra that are, in a certain sense, compatible with each other. Furthermore, they are supplemented with integrability conditions that are strong enough for useful results to hold while still leaving enough room for a diversity of solutions to exist. Among the central topics of Kähler geometry are Hodge theory and implications for complex algebraic varieties.

The more restrictive Calabi-Yau manifolds, which we will define as compact Kähler manifolds that can be equipped with a compatible complex volume form, have garnered particular interest: Supersymmetric string theory requires them in order to be compatible with observation and motivates various conjectures through physical principles, such as *mirror symmetry.*

The Calabi-Yau theorem roughly states that it is possible to adjust a Kähler structure on a compact manifold in a unique way to realize any reasonable Ricci tensor. While it is not constructive and only states the unique existence of such an adaptation, it is highly useful both in proving other theorems and by allowing one to find whole classes of examples of various specific structures, such as Calabi-Yau manifolds.

In this thesis, we will be taking a structural approach to introduce these topics and try to give an overview of how the various relevant components of Kähler manifolds relate to each other. To this end, we will use the language of G-structures and explore in particular intrinsic torsion and integrability. We will introduce each structure separately, roughly in the order of increasing restrictivity, and state results in their natural context. A number of theorems and tools that are central to the study of each structure but not of major importance to the topics above will be mentioned in passing, at least.

The remainder of this thesis is divided into three chapters: We will begin by introducing connections on fiber bundles as well as some foundational concepts and results. We then go on to systematically present the various layers of structures that make up Kähler and Calabi-Yau manifolds. Finally, we will focus on the Calabi-Yau theorem in particular to discuss its implications, reformulate it as a differential equation and give a rough outline of its proof.

Conventions and Prerequisites

In the following, a manifold is assumed to be a second-countable Hausdorff space with a differentiable structure, i.e. an equivalence class of atlases of charts to $\mathbb{R}^n$ with differentiable transition functions. It comes equipped with a tangent bundle TM giving rise to the $(r,s)-$tensor bundles $T^r_sM = \bigotimes^r TM \otimes \bigotimes^s T^*M$ and the k-form bundle $\Lambda^k M := \Lambda^k T^*M$, whose smooth sections are the differential forms in $\Omega^k M = \Gamma^\infty(\Lambda^k M)$. We use the Einstein convention, i.e. repeated indices are implicitly summed over. The de Rham cohomology groups $H^k_{dR}(M,\mathbb{K})$ where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ arise from the cochain complex that has the exterior derivative as coboundary map and yield Betti numbers $b_k(M) := \dim_{\mathbb{K}} H^k_{dR}(M,\mathbb{K})$. We embed the exterior algebra into the tensor algebra in such a way that

$$(\alpha \wedge \beta)(X_1, ..., X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \operatorname{sgn} \sigma \; \alpha\left(X_{\sigma(1)}, ..., X_{\sigma(k)}\right) \beta\left(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)}\right) \tag{1}$$

holds for $\alpha \in \Omega^k M, \beta \in \Omega^l M$ and tangent vectors X_j at the same point.

We will assume the theory of differential manifolds, Lie groups, fiber bundles (in particular principal and vector bundles) and de Rham cohomology. For the first real Chern class of a complex manifold M, we will only need the basic result that it is represented by the form $\underline{c}_1 := -\operatorname{Tr}^{\mathbb{C}}(R)/2\pi i$ where R is the curvature of any connection on M. Furthermore, the next chapter will summarize some aspects of the theory of connections on fiber bundles but only prove selected results.

We will also refrain from reproducing some elaborate proofs in the main text, but always give references to a full account. This mainly applies to the theorems of Moser, Darboux and Newlander-Nirenberg as well as the Hodge decomposition, Kähler identities and the full details of Yau's proof of the Calabi conjecture.

2 Connections on fiber bundles

This chapter is meant to concisely introduce connections on bundles and summarise the results necessary for the remainder of this thesis. For this reason, many theorems will be stated without proof here. More thorough expositions can for example be found in [1, 2, 3, 4, 5], on which the following is partly based.

The first section starts by introducing the general notion of connections on fiber bundles and discusses the special cases of principal, linear and manifold connections. The next section defines curvature and torsion as central invariants associated with connections. Finally, we discuss G-structures as a way to systematically define structures on manifolds as well as their interaction with connections.

The glaring omission is the discussion of parallel sections and transport, which leads to holonomy as another very important invariant.

2.1 Connections

2.1 Definition. Let $\pi : E \to M$ be a smooth fiber bundle. $\mathcal{V}_p := \ker d\pi_p$ defines a distribution, the so-called **vertical bundle**.

A **connection** on E is characterized by a **horizontal distribution** $\mathcal{H}$ so that $TE = \mathcal{V} \oplus \mathcal{H}$. Equivalently, it can be described through a **connection form** $\omega : TE \to \mathcal{V}$ with $\omega^2 = \omega$ and $\omega|_{\mathcal{V}} = \mathrm{Id}_{\mathcal{V}}$, which is just the associated projection map so that $\mathcal{H}_p = \ker \omega_p$.

A connection evidently *connects* the fibers close to each other by giving a sense of direction in which one can move in E purely horizontally without moving within the fiber. For fiber bundles with more vertical structure, we find natural compatibility conditions for this interpretation to make sense:

2.2 Def. & Prop. Let $\pi : P \to M$ be a principal G-bundle over a manifold M. A connection on P is called a **principal connection** if $\mathcal{H}$ is right-invariant under the vertical Lie structure, i.e.

$$\mathcal{H}_p \cdot g = \mathcal{H}_{p\cdot g} \qquad \forall p \in P, g \in G. \tag{2}$$

We can canonically identify each $\mathcal{V}_p$ with $\mathfrak{g}$ by mapping the left-invariant vector fields $X \in \mathfrak{g}$ to the fundamental vector field X_* evaluated at $p \in P$:

$$(X_*)_p := \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp tX. \tag{3}$$

This allows us to interpret the connection form ω as an element of $\Omega^1(P, \mathfrak{g})$ so that

$$\omega(X_*) = X \quad \text{and} \quad R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega. \tag{4}$$

2.3 Def. & Prop. Let $\pi : E \to M$ be a vector bundle over a manifold M with typical fiber V. A connection on E is called a **linear connection** if it is right-invariant under the canonical action of the general linear group of the fiber:

$$\mathcal{H}_p \cdot g = \mathcal{H}_{p \cdot g} \qquad \forall p \in P, g \in GL(V). \tag{5}$$

Equivalently, a linear connection can be described through a **covariant derivative**, which is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ so that the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma \quad \forall f \in C^\infty(M), \sigma \in \Gamma(E) \tag{6}$$

holds. This map describes the projection onto the vertical fiber that a section of E exhibits when moving along a tangential vector on the base manifold. For such a $X \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$ we also write $\nabla_X\sigma := (\nabla\sigma)(X) \in \Gamma(E)$.

2.4 (Existence and multiplicity)**.** Given any fiber, principal or vector bundle $\pi : F \to M$ we can always define a compatible connection: Considering appropriate local trivializations over an open cover, we can use a partition of unity to combine the flat connections associated with each trivialization, i.e. sewing together the kernels of the local projections onto the model fiber to form a horizontal distribution.

The difference of two connection forms ω_1 and ω_2 on E clearly is an idempotent form α that vanishes on the vertical bundle. If $\omega_{1/2}$ are principal or linear connections, then α is also invariant under the respective right-action. Indeed, the set of all possible compatible connections on a manifold M is an affine space with the space of such α as linear part. For principal connections, this can be identified with the space $\Omega_{\mathrm{Ad,hor}}(F, \mathfrak{g})$ of Ad-equivariant horizontal forms with values in the Lie algebra.

For vector bundles that arise through a principal bundle and a vector space representation of its Lie group, any principal connection induces a linear connection and the covariant derivative can be reexpressed:

2.5 Def. & Prop. Let $\pi : P \to M$ be a principal G-bundle over a manifold M and $\rho : G \to GL(V)$ a representation of G on a vector space V. Any principal connection on P reduces to a linear connection on the associated vector bundle $E := P \times_{G,\rho} V$. There is a canonical identification of the space $\Omega^k_{\rho,\mathrm{hor}}(P, V)$ of V-valued horizontal $\rho-$equivariant k-forms on P with $\Omega^k(M, E)$. This allows us to interpret the covariant derivative as the **covariant differential**

$$\begin{aligned} D_\omega : \Omega^k_{\rho,\mathrm{hor}}(P, V) &\to \Omega^{k+1}_{\rho,\mathrm{hor}}(P, V) \\ \alpha &\mapsto D_\omega\alpha := \left((v_0, ..., v_k) \mapsto d\alpha(v_0^h, ..., v_k^h)\right), \end{aligned}$$

where the superscript h refers to the projection of the tangent vector to the horizontal bundle.

In the case of the frame bundle, this correspondence is bijective so that we can define:

2.6 Definition. A **connection on a manifold** M is a linear connection on the tangent bundle, or, equivalently, a principal connection on its frame bundle.

2.2 Curvature and Torsion

2.7 Def. & Prop. Let $\pi : E \to M$ be a vector bundle over M equipped with a linear connection ∇. There then exists a unique $R \in \Omega^2(M, \mathrm{End}\, E)$, called the **Riemann curvature**, so that

$$R(X \wedge Y)\,\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma \qquad \forall X, Y \in \Gamma(TM), \sigma \in \Gamma(E). \tag{7}$$

The curvature satisfies the *differential Bianchi identity*

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0. \tag{8}$$

If E arises as an associated bundle from a representation ρ on a principal G-bundle P with a principal connection ω, the Riemann curvature can be seen as arising from the **Curvature form** $\Omega := D_\omega \omega \in \Omega^2(M, \operatorname{Ad} P) \simeq \Omega^2_{\mathrm{Ad,hor}}(P, \mathfrak{g})$ via $R = \rho_* \Omega$. The differential Bianchi identity then asserts

$$D_\omega \Omega = 0. \tag{9}$$

2.8 (Curvature as an obstruction to integrability)**.** One can consider a connection on a fiber bundle to be **integrable** if the connection form can locally be regarded as the differential of the vertical projection map of a local trivialization. It turns out that non-vanishing curvature is exactly the obstruction to this sense of integrability on principal and vector bundles: $\Omega = 0$ and $R = 0$, respectively, are equivalent to the existence of such local trivializations as well as the integrability of the horizontal distribution. In these cases we speak of a **flat** connection.

We will later have to consider line bundles in particular:

2.9 (Connections and curvature on line bundles)**.** Multiplications with a scalar are the only linear maps on a one-dimensional space. This means that a linear connection ∇ on a line bundle can locally be expressed as a 1-form η with values in the respective field: Given a local frame σ, which is just a non-vanishing local section in this case, we define for any tangent vector X

$$\eta(X)\,\sigma := \nabla_X \sigma. \tag{10}$$

Any other section can be expressed relatively to σ with a function f as $f\sigma$. $\nabla f\sigma$ is then already determined by the preceding equation. By writing out the definition one immediately sees that the curvature tensor associated with ∇ is locally given by

$$R(X \wedge Y)\,\sigma = d\eta(X,Y)\,\sigma. \tag{11}$$

It is straightforward to see that the curvature vanishes in a region if and only if a local non-vanishing parallel section exists: By the last equation and the Poincaré Lemma, $R = 0$ implies $\eta = dg$ for a local function g. We can then check that $e^{-g}\sigma$ is a local parallel section:

$$\nabla_X(e^{-g}\sigma) = X(e^{-g})\,\sigma + e^{-g}\eta(X)\,\sigma = (-X(g) + dg(X))\;e^{-g}\sigma = 0. \tag{12}$$

The other direction is trivial. This is of course just a concrete example of the general principle mentioned in 2.8.

Given a connection *on a manifold*, the fact that the covariant derivative of a vector field is another vector field allows us to define a reduced notion of curvature and another invariant that relates to a stronger concept of integrability:

2.10 Def. & Prop. Let M be a manifold equipped with a connection. The **Ricci tensor** $\operatorname{Ric} \in T^0_2 M$ is then defined as the contraction

$$\operatorname{Ric}(X,Y) := \operatorname{Tr}(Z \mapsto R(Z,X)Y). \tag{13}$$

There also exists a unique $T \in \Omega^2(M, TM)$, called the **Torsion**, so that

$$T(X \wedge Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad \forall X, Y \in \Gamma(TM). \tag{14}$$

Since TM is associated to the frame bundle $\mathcal{F}M$ over M, T can be regarded as an element of $\Omega^2_{GL(\mathbb{R}^n),\mathrm{hor}}(\mathcal{F}M, \mathbb{R}^n)$ and is then referred to as the **Torsion form** Θ.

2.11 (Torsion as an obstruction to integrability)**.** We call a connection *on a manifold* M **integrable**, if one can find a chart around every point of M so that its coordinate frame is horizontal. It turns out that this is equivalent to the vanishing of *both* curvature and torsion. It also clearly implies the weaker notions of integrability of the principal connection on the frame bundle and of the linear connection on the tangent bundle.

2.12 (Identities of torsion-free connections)**.** Given a torsion-free connection on a manifold M, we can find the *algebraic Bianchi identity*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{15}$$

which implies that the Ricci tensor is symmetric. Moreover, the Lie bracket can in this case be expressed through the covariant derivative as $[X,Y] = \nabla_X Y - \nabla_Y X$.

2.13 (Solder forms)**.** Let $\pi : E \to M$ be an n-dimensional vector bundle and $\mathcal{F}E$ its frame bundle, which is equipped with a principal connection ω. A **solder form** $\theta \in \Omega(M,E) \simeq \Omega_{GL(\mathbb{R}^n),\mathrm{hor}}(\mathcal{F}E,\mathbb{R}^n)$ allows a more general definition of torsion as $\Theta = D_\omega\theta$. The algebraic Bianchi identity then takes the form

$$D_\omega\Theta = \Omega \wedge \theta. \tag{16}$$

The usual definitions are recovered when equipping the tangent bundle of a manifold with its identity as the canonical solder form.

2.3 G-Structures

2.14 Definition. Let M be an n-manifold so that its frame bundle $\mathcal{F}M$ is a principal bundle with fiber $GL(n,\mathbb{R})$. For an embedded (regular) Lie subgroup $G \hookrightarrow GL(n,\mathbb{R})$ we define a **G-structure** to be a principal G-bundle P over M with a G-equivariant fiber bundle inclusion into $\mathcal{F}M$.

Effectively, this just means that we smoothly choose preferred sets of bases in the tangent spaces. As we will see in the next chapter, a large number of structures on manifolds can be described as G-structures for a suitable G. We will in particular consider closed subgroups of $GL(n,\mathbb{R})$, which by Cartan's closed-subgroup theorem automatically are embedded regular Lie subgroups.

2.15. Let M be a manifold and G a closed subgroup of $H := GL(\dim M,\mathbb{R})$. We can then define a bijective correspondence between sections of the fiber bundle

$$\mathcal{F}M/G := \mathcal{F}M \times_H H/G \tag{17}$$

and G-structures on M by mapping $\sigma \in \Gamma(\mathcal{F}M/G)$ to the subbundle $\pi^{-1}_{\mathcal{F}M/G}(\sigma(M))$ where $\pi_{\mathcal{F}M/G} : \mathcal{F}M \to \mathcal{F}M/G$ is the projection. Clearly, such sections and therefore G-structures do not necessarily need to exist globally.

For a given G-structure, there are natural compatibility conditions with connections and charts, which in turn yield notions of integrability:

2.16 Definition. A connection on a manifold M is called **compatible** with a G-structure P on M, if the principal connection on $\mathcal{F}M$ reduces[1] to a principal connection in the subbundle P. A coordinate chart on a manifold M is called **compatible** if the induced local frames are also sections of P.

A G-structure on a manifold M is called **(intrinsically) flat** or **torsion-free** if there exists a compatible connection with vanishing curvature or torsion tensor, respectively. It is **integrable** if the manifold can be covered with compatible coordinate charts.

For structures defined through an invariant tensor, these conditions take a particularly simple form:

2.17 Proposition. Let K_0 be a model tensor over $\mathbb{R}^n$ and $G \subseteq GL(n, \mathbb{R})$ the group of linear transformations that leave K_0 invariant, i.e. its (closed!) stabilizer. We then find:

(a) A G-structure on an n-dimensional manifold M is equivalent to the choice of a tensor K on M that is pointwise similar to K_0. The bundle of such tensors is isomorphic to $\mathcal{F}M/G$ from 2.15.

(b) A chart on M is compatible if and only if the coefficients of K relative to that chart are given by K_0.

(c) A connection on M is compatible with the G-structure precisely when $\nabla K = 0$.

Proof.
(a): For a G-structure P we can choose a frame $u \in P_x$ for every $x \in M$, which maps the fiber over x of any tensor bundle onto the respective set of tensors on $\mathbb{R}^n$. We can then define a tensor K by pulling back K_0. This is independent of the choice of frame since any other frame in P is related to u via a transformation in G, which leaves K_0 invariant. Conversely, given a tensor K that is pointwise similar to K_0, the frames in which it takes the form of K_0 assemble into a principal G-bundle. By construction, this also gives a bundle isomorphism to $\mathcal{F}M/G$.

(b) follows immediately since $u \in P$ is equivalent to saying that K takes the form K_0 relative to the frame u.

(c): Given a compatible connection, K can be seen to arise from an equivariant 0-form over TP with values in the appropriate tensor bundle of $\mathbb{R}^n$. $\nabla K = 0$ then holds exactly when the exterior derivative of this 0-form vanishes, i.e. when it is constant. But this is the case per definition, since according to (a), K takes the form K_0 relative to all frames in P. If, conversely, the connection was not compatible we could find a horizontal tangent vector X in the frame bundle that is not tangent to P, implying that $\nabla_{\pi_* X} K \neq 0$.

□

Note that (b) in particular implies that a G-structure is integrable if and only if it can be covered with coordinate charts such that K has constant coefficients, since these charts are compatible up to a linear transformation in $GL(n, \mathbb{R})/G$.

2.18 Def. & Prop. Let M be a manifold equipped with a G-structure P and let $V := \mathbb{R}^n$. We define the map

$$\sigma : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^* \tag{18}$$

as the inclusion $\mathfrak{g} \subseteq V \otimes V^*$ followed by antisymmetrization in the covariant indices. The exact sequence[2]

$$0 \to \ker \sigma \xrightarrow{\iota} \mathfrak{g} \otimes V^* \xrightarrow{\sigma} V \otimes \Lambda^2 V^* \xrightarrow{\text{pr}} \operatorname{coker} \sigma \to 0 \tag{19}$$

then yields, via the natural representations of G on each of these spaces, vector bundles associated with the G-structure:

$$0 \to \ker \tilde{\sigma} \xrightarrow{\iota} \operatorname{Ad} P \otimes T^*M \xrightarrow{\tilde{\sigma}} TM \otimes \Lambda^2 T^*M \xrightarrow{\text{pr}} \operatorname{coker} \tilde{\sigma} \to 0. \tag{20}$$

The difference between two covariant derivatives ∇ and ∇' on TM that are compatible with P yields a section of $\operatorname{Ad} P \otimes T^*M$. Eq. (14) then directly implies that the difference between the associated torsions T and T' is given by $\sigma(\nabla - \nabla') \in \Gamma(TM \otimes \Lambda^2 T^*M)$.

[1] Note that this is equivalent to either the connection form taking values in $\mathfrak{g}$ when restricted to TP or the horizontal bundle being tangent to P. For the covariant derivative of any associated vector bundle, this means $\nabla_X(g\sigma) = g\nabla_X \sigma \quad \forall g \in G$.

We therefore define the **intrinsic torsion** $T^i(P) = \mathrm{pr}(T) \in \Gamma(\operatorname{coker}\tilde{\sigma})$ of the G-structure through the torsion tensor T of any compatible connection. By the above, this is independent of the choice of compatible connection and vanishes everywhere exactly if a torsion-free connection compatible with the G-structure exists. Moreover, the space of torsion-free connections compatible with P is isomorphic to $\Gamma(\ker\tilde{\sigma})$.

2.19 (Intrinsic torsion as an obstruction to integrability of a G-structure)**.** An integrable G-structure on a manifold M is always free of intrinsic torsion: Similarly to the argument in 2.4, we can use compatible charts to locally define compatible connections. Evaluating their torsion on the induced basis shows that it indeed vanishes. Since the torsion of the sewn connection decomposes into those of the constituent local connections, it vanishes as well.

An integrable G-structure does not, however, need to be free of intrinsic curvature. An approach similar to the above fails because the curvature tensor does not simply decompose into the curvature tensors of the local connections as it is a second-order invariant.

We will see in the next chapter that, in many cases, a structure without intrinsic torsion is already integrable. A simple example of this are distributions: They can be regarded as G-structures where G is the subgroup of $GL(n,\mathbb{R})$ that leaves the distribution invariant. The Frobenius Theorem asserts that the vanishing of the Frobenius tensor, which can be identified with the intrinsic torsion, implies integrability of the distribution.

[2] The kernel of σ is also called the *first prolongation* $\mathfrak{g}^{(1)}$ and the cokernel is denoted $H^{0,1}(\mathfrak{g})$. This is part of the Spencer cohomology that follows from the cochain complex $C^{p,q} := \mathrm{Sym}^p V^* \otimes \Lambda^q V^*$ together with a boundary map that can be interpreted as the exterior derivative of polynomial forms. For more, see e.g. [6].

3 Layers of structure on Kähler manifolds

The group $U(n)$ can be regarded as a subgroup of a number of groups, each of it interesting in its own right:

$$U(n) = O(2n) \cap Sp(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) \tag{21}$$

This richness is paralleled in Kähler manifolds, which are just the manifolds equipped with a torsion-free $U(n)$-structure. The purpose of this chapter is to peel off the different layers of this structure to explore their interrelations and put it back together as a consistent whole.

We will begin with a quick discussion of orientations, volume forms, Riemannian metrics and symplectic forms on manifolds from the perspective of structure group reductions. We then introduce (almost) complex structures and explore the splitting of complexified tangent and differential form bundles as well as their integrability. Hermitian and Kähler manifolds then arise naturally as a combination of the above structures and we will particularly discuss their Hodge theory. Finally, we introduce Calabi-Yau manifolds and give a brief overview of analogous constructions based on the quaternionic general linear group.

3.1 Oriented manifolds as $GL^+(n, \mathbb{R})$-structures

3.1 Definition. An **Orientation** on an n-manifold M is a $GL^+(n, \mathbb{R})$-structure.

3.2 (Alternative description)**.** This is equivalent to a smooth choice of vector space orientations on each of the tangent spaces in TM, where the reduced frame bundle P contains exactly the positively-oriented bases. These orientations are compatible, since by definition the transition maps have a positive determinant.

3.3 (Existence and multiplicity)**.** According to 2.15, an orientation is a section of $\tilde{M} := \mathcal{F}M/GL^+(n, \mathbb{R})$, which carries the fiber $GL(n, \mathbb{R})/GL^+(n, \mathbb{R}) \cong \mathbb{Z}_2$. If $\tilde{M}$ is connected, then it has to be a double cover of M without global sections so that no orientations exist. If it is not connected, $\tilde{M}$ is a trivial bundle with exactly two possible global sections to choose from.

From the point of view of characteristic classes, a manifold is orientable if and only if its first Stiefel-Whitney class is zero.

3.4 (Compatible charts and connections)**.** Any connection on the manifold is compatible since $GL^+(n, \mathbb{R})$, as a connected component of $GL(n, \mathbb{R})$, carries the same Lie algebra and any coordinate chart can be made compatible by reversing its orientation if necessary. In particular, this implies that a $GL^+(n, \mathbb{R})$-structure is automatically integrable and thereby also free of intrinsic torsion.

3.2 Volume forms as $SL(n, \mathbb{R})$-structures

3.5 Definition. A **volume form** on an n-manifold M is a $SL(n, \mathbb{R})$-structure.

3.6 (Alternative description)**.** As the name suggests, this definition is equivalent to the choice of a non-vanishing global section μ of the canonical line bundle $\Omega^n(M)$. This follows as discussed in 2.17 since $SL(n,\mathbb{R})$ can be defined as the stabilizer of the standard volume form on $\mathbb{R}^n$.

3.7 (Existence and multiplicity)**.** A volume form exists if and only if M is orientable: On one hand, a volume form induces an orientation on M since $SL(\dim M,\mathbb{R}) \subseteq GL^+(\dim M,\mathbb{R})$. On the other hand, an orientation on M induces an orientation on the canonical line bundle, which already implies its triviality since its fiber is $\mathbb{R}$. Locally, we can of course always find a volume form. Some global features on non-orientable manifolds can be salvaged by considering densities instead.

For orientable M, the space of volume forms is an affine space with the space of everywhere non-vanishing real functions over M as its linear part since any two volume forms μ, μ' are related by exactly one such f so that $\mu' = f\mu$.

3.8 (Induced measure)**.** Given a volume form μ on a manifold M, there exists exactly one measure on the Borel sets that we denote vol_μ so that the volume of any open subset U is given by

$$\mathrm{vol}_\mu(U) := \int_U \mu \tag{22}$$

and functions $f \in C^\infty(M)$ are integrated as

$$\int_U f \; d\mathrm{vol}_\mu = \int_U f \cdot \mu \tag{23}$$

The volumes of compact connected components of M are global invariants associated with the volume form. These are indeed the only independent invariants due to the following result:

3.9 Theorem (Moser [7])**.** Given two compact connected diffeomorphic manifolds M and N that are equipped with volume forms μ and ν, there exists a diffeomorphism mapping M to N and μ to ν if and only if $\int_M \mu = \int_N \nu$.

This also allows us to easily construct compatible charts so that any $SL(\dim M,\mathbb{R})$-structure is both integrable and free of intrinsic torsion.

3.3 Riemannian manifolds as $O(n)$-structures

3.10 Definition. A **Riemannian structure** on an n-manifold M is an $O(n)$-structure.

3.11 (Alternative description as a metric tensor)**.** Since the orthogonal group is just the stabilizer of the standard scalar product on $\mathbb{R}^n$, 2.17 and Sylvester's law of inertia allow us to identify a Riemannian structure with a symmetric and positive definite $(0,2)$-tensor g, giving an inner product on the tangent spaces.

The metric can also be seen as a linear map $\varphi_g : TM \to T^*M, v_p \mapsto g_p(v_p, \cdot)$, which is an isomorphism since g is non-degenerate. It is referred to as the **musical isomorphism** because the shorthands $v^\flat := \varphi_g(v)$ and $\alpha^\sharp := \varphi_g^{-1}(\alpha)$ raise and lower indices in local coordinates.

The metric induces an inner product $\varphi_g^{-1\,*}g$ on the cotangent spaces, which we will also denote g, and extends to the fibers of the tensor and form bundles as well. We will need the latter in particular, which is determined by

$$g(\alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k) = \det(g(\alpha_i, \beta_j)) \tag{24}$$

for 1-forms α_j and β_j.

3.12 (Existence)**.** A Riemannian structure can be defined on any manifold M by pulling back the standard metric along a locally finite set of charts and gluing them together with the associated partition of unity.

3.13 (Compatible connections and intrinsic torsion)**.** By 2.17, compatible connections are those with $\nabla g = 0$. Following [8], the first prolongation $\mathfrak{o}(n)^{(1)} = \ker\sigma$ from 2.18 equals 0, since for any $\alpha \in \mathfrak{o}(n) \otimes (\mathbb{R}^n)^*$ we find

$$\begin{aligned} g(\alpha(x)y, z) \;&=\; -g(y, \alpha(x)z) \;=\; -g(y, \alpha(z)x) \;=\; g(\alpha(z)y, x) \\ &=\; g(\alpha(y)z, x) \;=\; -g(z, \alpha(y)x) \;=\; -g(\alpha(x)y, z) \qquad \forall x, y, z \in \mathbb{R}^n, \end{aligned}$$

where we have used skew-symmetry of $\alpha(X)$ and $\sigma(\alpha) = 0$. On dimensional grounds, this already implies that σ is an isomorphism and so there always exists exactly one torsion-free compatible connection, the **Levi-Civita connection** ∇^g. In particular, any $O(n)$-structure is free of intrinsic torsion. A more direct characterization of the Levi-Civita connection is given by

$$\begin{aligned} 2\, g(\nabla^g_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) + Z(g(X,Y)) \\ + g([X,Y],Z) + g([Z,X],Y) + g(X,[Z,Y]), \end{aligned} \tag{25}$$

which readily follows from $\nabla^g g = 0$ and $T^g = 0$.

3.14 (Curvature of the Levi-Civita connection and Integrability)**.** By $\nabla R = 0$ we find the additional symmetry relation

$$g(R(X,Y)\,U,V) + g(R(X,Y)\,V,U) = 0 \tag{26}$$

of the curvature tensor of the Levi-Civita connection, which in particular implies the symmetry $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$ of the Ricci tensor. We can also define the scalar curvature Scal by contracting the remaining indices of Ric using the metric.

Moreover, the Levi-Civita curvature is the only obstruction to integrability, i.e. the Riemannian structure is integrable if and only if $R = 0$ everywhere (see e.g. 10.6.7 of [9]).

3.15 (Einstein manifolds)**.** An **Einstein manifold** is a Riemannian manifold such that the Ricci tensor is proportional to the metric:

$$\mathrm{Ric} = \lambda\, g. \tag{27}$$

This in particular implies $\mathrm{Scal} = \lambda\, \dim M$.

There is, of course, a large amount of results related to the induced metric space structure as well as the exponential map and geodesics. This is outside of the scope of this short summary, however.

Oriented Riemannian manifolds and Hodge Theory

3.16 (Oriented Riemannian manifolds)**.** An **oriented Riemannian structure** on an n-manifold M is an $SO(n)$-structure. As the simultaneous stabilizer of the standard metric and volume form, it is given by a Riemannian structure together with a volume form that can be written

$$\mu_g = \sqrt{\det(g(\partial x_i, \partial x_j))}\; dx^1 \wedge \ldots \wedge dx^n \tag{28}$$

in positively oriented local coordinates $x^1, \ldots, x^n$. Any Riemannian structure can locally be oriented by considering a simply-connected neighbourhood.

An oriented Riemannian structure gives us additional useful tools:

3.17 (Hodge star and L^2-inner product)**.** We can define the **Hodge star** operator as the unique linear bundle map $* : \Lambda^k M \to \Lambda^{n-k} M$ that satisfies

$$\alpha \wedge (*\beta) = g(\alpha, \beta)\mu_g \tag{29}$$

for two k-forms α and β, where the metric on the right is to be understood as in Eq. (24). Existence and uniqueness of such a map can easily be seen by writing this out in a positively oriented orthonormal basis. This also yields the following consequences of the definition:

$$*\,1 = \mu_g, \qquad **\,\alpha = (-1)^{k(n-k)}\alpha, \qquad \text{and} \qquad g(\alpha, \beta) = g(*\alpha, *\beta), \tag{30}$$

where α, β are elements of $\Omega^k(M)$ again.

We can also define the **L^2-inner product** on the space $\Omega_c^k(M)$ of k-forms with compact support:

$$< \alpha, \beta >_{L^2} := \int_M g(\alpha, \beta)\, d\mu_g = \int_M \alpha \wedge (*\beta). \tag{31}$$

3.18 (Codifferential and Laplacian)**.** We define the **codifferential** $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ as

$$d^* := (-1)^{kn+n+1} \; * \circ\, d \circ * \tag{32}$$

for $k > 0$ and $d^* f = 0$ for functions $f \in \Omega^0(M)$. Integration by parts using Stokes' Theorem shows that this is the formal adjoint to the exterior derivative with respect to the L^2-inner product, i.e. $< \alpha, d\beta >_{L^2} = < d^*\alpha, \beta >_{L^2}$. One easily sees $(d^*)^2 = 0$ and we say α is **coclosed** if $d^*\alpha = 0$ and **coexact** if there exists a β such that $\alpha = d^*\beta$.

The **Hodge-Laplacian** $\Delta_d : \Omega^k(M) \to \Omega^k(M)$ is defined as

$$\Delta_d = dd^* + d^*d = (d + d^*)^2. \tag{33}$$

Hodge theory is essentially the study of this operator. A k-form α is called **harmonic** if $\Delta_d\,\alpha = 0$. One can see by considering $0 =< \alpha, \Delta_d\alpha >_{L^2}$ that this is equivalent to being both closed and coclosed. We write $\mathcal{H}_d^k(M) := \ker(\Delta_d : \Omega^k(M) \to \Omega^k(M))$ for the space of such forms.

The following central result gives particular importance to harmonic forms:

3.19 Theorem (Hodge Decomposition Theorem)**.** Let M be a compact and oriented Riemannian manifold. The space of k-forms decomposes into

$$\Omega^k(M) = \mathcal{H}_d^k \oplus d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M), \tag{34}$$

and the three constituent spaces are orthogonal with respect to the L^2-inner product.

The elaborate proof of the direct sum decomposition can be found e.g. as 6.8 of [10]. Orthogonality on the other hand is just an immediate consequence of the construction of d^* as formal adjoint. Reaping the rewards of this Theorem, we find:

3.20 Corollary (Hodge's Theorem)**.** Let M be a compact and oriented Riemannian n-manifold. Every element of the k-th de Rham cohomology group contains exactly one harmonic k-form so that $H_{dR}^k(M)$ is naturally isomorphic to $\mathcal{H}_d^k(M)$.

Proof. The statements follow if $\phi : \alpha \mapsto [\alpha]$ is an isomorphism from $\mathcal{H}_d^k(M)$ to $H_{dR}^k(M)$. It is well-defined since we have already seen that harmonic forms are closed. ϕ is injective since two harmonic forms in the same de Rham class have to differ by an element of $d\Omega^{k-1}(M)$ that is also harmonic, so by the Hodge decomposition the difference can only be zero. Finally,

ϕ is surjective: A closed form α representing a given de Rham class can be decomposed into $\alpha = \alpha^H + d\beta + d^*\gamma$, where α^H is harmonic. Since α is closed,

$$0 =< d\alpha, \gamma >_{L^2} =< dd^*\gamma, \gamma >_{L^2} = ||d^*\gamma||_{L^2} \quad \Rightarrow \quad d^*\gamma = 0 \tag{35}$$

Then, $\phi(\alpha^H) = [\alpha^H] = [\alpha^H + d\beta] = [\alpha]$. □

3.21 Corollary (Poincaré duality)**.** Let M be a compact and oriented Riemannian n-manifold. The spaces $\mathcal{H}_d^k(M)$ and $\mathcal{H}_d^{n-k}(M)$ are naturally isomorphic and the Betti numbers satisfy $b_k(M) = b_{n-k}(M)$.

Proof. The isomorphism is simply given by the Hodge star, which we have already seen to be bijective. It maps harmonic forms onto harmonic forms because it commutes with Δ_d. The previous corollary then immediately gives the relationship between Betti numbers. □

3.4 Complex manifolds as $GL(n/2, \mathbb{C})$-structures

3.22 Definition. An **almost complex structure** on a manifold M of even dimension n is a $GL(n/2, \mathbb{C})$-structure.

3.23 (Alternative description)**.** The complex linear group $GL(n/2, \mathbb{C}) \subseteq GL(n, \mathbb{R})$ can be defined as the stabilizer of the endomorphism J_0 over $\mathbb{R}^n$ that is represented by the block matrix

$$J_0 = \begin{pmatrix} 0 & \mathrm{Id}_{n/2} \\ -\mathrm{Id}_{n/2} & 0 \end{pmatrix}, \tag{36}$$

since J_0 can be interpreted as multiplication with the imaginary unit in $\mathbb{C}^{n/2}$. A section J of the endomorphism bundle is pointwise similar to J_0 if and only if

$$\forall x \in M: \quad J_x^2 = -\mathrm{Id}_{T_x M} \tag{37}$$

because any such real J has paired eigenvalues $\pm i$ in the complexified vector space that can be rotated to individual $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ Jordan-blocks.

By 2.17, an almost complex structure is therefore equivalent to the choice of such a J on M. Note that it is not necessary to demand even dimensionality since this is already implied by $0 < (\det J_x)^2 = (-1)^n$.

3.24 (Orientation and Existence)**.** Since $GL(n/2, \mathbb{C}) \subseteq GL^+(n, \mathbb{R})$, almost complex structures carry a natural orientation. Even dimension and orientability are not sufficient for existence, however, since there can be further topological obstructions. We will see in 3.57 that existence of almost complex, almost symplectic and Hermitian structures are equivalent and discuss these obstructions in more detail there.

Complex vector bundles on almost complex manifolds

There are various ways in which we can construct complex vector bundles from the real tangent bundle TM:

3.25 Definition. Let M be an almost complex manifold. We define...

- the **complexified tangent bundle** $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$,
- the **complexified k-form bundle** $\Lambda^k_{\mathbb{C}}M := \Lambda^k M \otimes \mathbb{C}$,
- the **complexified tensor bundle** $T^r_{s\,\mathbb{C}}M := T^r_s M \otimes \mathbb{C}$, and

- the **holomorphic tangent bundle** $T_H M$ which is the complex vector bundle that arises from equipping the real tangent bundle TM with the complex vector space structure induced by J on every fiber.

While the first three bundles can be defined without reference to an almost complex structure, they are often of interest due to their interaction with it. Note that the artificial construction of a complex structure on $T_{\mathbb{C}}M$ doubles its real dimension so that

$$n = \dim_{\mathbb{C}} T_{\mathbb{C}}M = \dim_{\mathbb{R}} TM = 2\dim_{\mathbb{C}} T_H M. \tag{38}$$

The almost complex structure induces a splitting of this larger bundle into two copies of the holomorphic tangent bundle:

3.26 Def. & Prop (Splitting of $T_{\mathbb{C}}M$)**.** Let M be an almost complex manifold. We define two subbundles of $T_{\mathbb{C}}M$ as eigenspaces of J by setting

$$T'M := \ker(J - i\,\mathrm{Id}) \quad \text{and} \quad T''M := \ker(J + i\,\mathrm{Id}). \tag{39}$$

It then follows that

$$T_{\mathbb{C}}M = T'M \oplus T''M, \tag{40}$$

where $X \mapsto \frac{1}{2}(X - iJX)$ is a bijective $\mathbb{C}$-linear bundle map and $X \mapsto \frac{1}{2}(X + iJX)$ is a bijective $\mathbb{C}-$*skew*linear bundle map from $T_H M$ to $T'M$ and $T''M$, respectively.

Proof. It is easily checked that the given bundle maps are well-defined, bijective and (skew-)linear. $\ker(J \pm i\,\mathrm{Id})$ therefore has constant dimension on all fibers so that it is a subbundle. The transition functions of $T_{\mathbb{C}}M$ necessarily preserve $T'M$ and $T''M$ and can therefore be decomposed into a direct sum of transition functions of these. □

The splitting extends to complex forms as well:

3.27 Def. & Prop (Splitting of $\Lambda^k_{\mathbb{C}}M$)**.** Let M be an almost complex manifold. Defining the (p,q)**-form bundle**

$$\Lambda^{p,q}_{\mathbb{C}}M := \Lambda^p(T'M)^* \otimes \Lambda^q(T''M)^* \tag{41}$$

yields a natural splitting

$$\Lambda^k_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^{p,q}_{\mathbb{C}}M \tag{42}$$

and write $\Omega^{p,q}_{\mathbb{C}}M := \Gamma^\infty(\Lambda^{p,q}_{\mathbb{C}}M)$ for the space of differential forms of type (p,q). The action of J on the dual spaces of $T'M$ and $T''M$ naturally extends to any (p,q)-form α as

$$J\alpha = i^{(q-p)}\alpha. \tag{43}$$

Proof. Let π' and π'' be the projections onto $T'M$ and $T''M$, respectively. To split a complex k-form α into its parts of definite type, just expand

$$\alpha(X_1, ..., X_k) = \alpha\big((\pi' + \pi'')X_1, ..., (\pi' + \pi'')X_k\big). \tag{44}$$

Collecting terms that contain π' exactly p times yields a uniquely determined $(p, k-p)$-form. □

Clearly, we could split the complexified tensor bundle in a very similar way, but this would require some elaborate notation.

3.28 (Conjugation)**.** The complexified bundles are not just complex vector bundles but also carry a canonical notion of real and imaginary parts of vectors. For this reason, the usual complex conjugation is well-defined. By the isomorphisms that map $T_{\mathbb{C}}M$ onto $T'M$ and $T''M$, it is clear that conjugation is a (real) isomorphism between the latter bundles.

3.29 (Associated forms)**.** It is straightforward to check that there is a one-to-one correspondence between J-invariant[3] symmetric bilinear forms $b \in T^0_{2\mathbb{C}}M$ and $(1,1)$-forms $\beta \in \Omega^{1,1}_{\mathbb{C}}M$ by setting

$$b(X,Y) = \beta(X, JY) \qquad \text{or equivalently} \qquad \beta(X,Y) = b(JX,Y) \tag{45}$$

for all tangent vectors X, Y in a given point.

Note that either of these forms is real or non-degenerate exactly when the other is. We will call β **positive** or **negative** if b is positive- or negative-definite, respectively.

Integrability

3.30 Definition. A **complex structure** on an n-manifold M is an *integrable* $GL(n,\mathbb{C})$-structure. This is equivalent to defining **complex manifolds** through a maximal atlas of holomorphic charts, i.e. homeomorphisms to $\mathbb{C}^{n/2}$, with biholomorphic transition functions.

To illustrate the different bundles on a complex manifold M and fix our notation, we consider the local frames induced by holomorphic charts:

3.31 (Tangent space frame induced by holomorphic chart)**.** A holomorphic chart induces complex coordinates $z_j = x_j + iy_j$, giving a real basis $\{\partial x_j, \partial y_j | j = 1 \ldots \frac{n}{2}\}$ of the (real) tangent bundle TM as well as inducing a map J_x that sends ∂x_i to ∂y_i. This is of course also a basis for the complexified tangent bundle $T_{\mathbb{C}}M$ if we allow complex coefficients. Rewriting this to

$$\partial z_j := \frac{1}{2}\left(\partial x_j - i\partial y_j\right) \quad \text{and} \quad \partial \bar{z}_j := \frac{1}{2}\left(\partial x_j + i\partial y_j\right), \tag{46}$$

we obtain bases of $T'M$ and $T''M$, respectively. These are natural generalizations of the Wirtinger derivatives and in part motivate our consideration of the complexified tangent bundle instead of only the holomorphic tangent bundle T_HM. Pulling the ∂z_j back along the bundle isomorphism $T_HM \to T'M$, defined in 3.26, we obtain the ∂x_j as the natural basis for T_HM. The ∂y_j are not necessary here since they are identified with $i\partial x_j$.

3.32 (Cotangent space frame induced by holomorphic chart)**.** The dual basis of $\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}_{\mathbb{C}}M \oplus \Lambda^{0,1}_{\mathbb{C}}M$ is given by

$$dz^j := dx^j + idy^j \quad \text{and} \quad d\bar{z}^j := dx^j - idy^j, \tag{47}$$

so that a $(p,q)-$form α can always be written as

$$\alpha = \sum a_{j_1 \ldots j_p, k_1 \ldots k_q}\, dz^{j_1} \wedge \ldots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_q}. \tag{48}$$

We can continue the exterior differential $\mathbb{C}$-linearly onto the complexified bundle. Note that the representation of Eq. (48) then clearly implies that $d\alpha$ is a linear combination of uniquely determined forms of the type $(p+1,q)$ and $(p,q+1)$.

3.33 (Local representation of complex tensors)**.** For any complex tensor $T \in \Gamma(T^r_{s\,\mathbb{C}}M)$, we will use Greek indices with or without a bar to refer to the local representation that arises from evaluating the tensor on ∂z_α or dz_α with or without a bar, respectively. For example, given a $T \in \Gamma(T^1_{1\,\mathbb{C}}M)$ we would write

$$T^{\alpha}{}_{\beta} = T(dz^\alpha, \partial z_\beta), \quad T^{\bar{\alpha}}{}_{\beta} = T(d\bar{z}^\alpha, \partial z_\beta), \quad T^{\alpha}{}_{\bar{\beta}} = T(dz^\alpha, \partial \bar{z}_\beta), \quad T^{\bar{\alpha}}{}_{\bar{\beta}} = T(d\bar{z}^\alpha, \partial \bar{z}_\beta), \tag{49}$$

where each of these matrices contains the components of T after projecting it into one of the subspaces of the splitting induced by J. Note that we do not regard the bar as part of the index but rather as part of the object it is attached to.

[3]Due to the naturally induced action of J on a bilinear form b this just means $b(X,Y) = b(JX,JY)\ \forall X,Y \in T_pM, p \in M$. This holds automatically for differential forms of type $(1,1)$.

On almost complex manifolds, we can only say that the exterior derivative of a (p,q)-form is a linear combination of (r,s)-forms where $r+s=p+q+1$ instead of the stronger result of 3.32. The fact that it takes a simple shape on complex manifolds is an expression of vanishing intrinsic torsion:

3.34 Proposition (Characterizations of torsion-free almost complex structures)**.** Let M be a manifold equipped with an almost complex structure. The following statements are equivalent:

(a) The almost complex structure is torsion-free.

(b) The **Nijenhuis tensor** $\mathrm{Nij}(X,Y)=[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY]$ vanishes for all $X,Y\in T_pM, p\in M$.

(c) The Lie bracket of two (anti-)holomorphic vector fields, i.e. sections of $T'M$ $(T''M)$, is itself (anti-)holomorphic.

(d) The exterior derivative of $(1,0)-$forms has no component of type $(0,2)$.

(e) The exterior derivative of $(p,q)-$forms only has components of type $(p+1,q)$ and $(p,q+1)$.

Proof.
(a) $\Longleftrightarrow$ **(b)**: This follows since the Nijenhuis tensor can be identified with the intrinsic torsion. To see this, note that any $\alpha\in\Gamma(TM\otimes\Lambda^2T^*M)$ in 2.18 decomposes into three tensors $\alpha=\alpha^++\alpha^-+\alpha^0$ defined by

$$\alpha^\pm(X,Y):=\frac{1}{4}\left(\alpha(X,Y)\mp J\alpha(JX,Y)\mp J\alpha(X,JY)-\alpha(JX,JY)\right), \tag{50}$$

$$\alpha^0(X,Y):=\frac{1}{2}\left(\alpha(X,Y)+\alpha(JX,JY)\right), \tag{51}$$

for all $X,Y\in\Gamma(TM)$ so that

$$J\alpha^\pm(X,Y)=\pm\alpha^\pm(JX,Y)=\pm\alpha^\pm(X,JY)\qquad\text{and}\qquad\alpha^0(X,Y)=\alpha^0(JX,JY). \tag{52}$$

Since preimages of any α^+ and α^0 are readily constructed, the image of $\tilde\sigma$ is given by those α with $\alpha^-=0$. For any compatible connection ∇, the projection of the torsion tensor T onto $\operatorname{coker}\tilde\sigma$ is then just

$$T^-(X,Y):=T(X,Y)+JT(JX,Y)+JT(X,JY)-T(JX,JY). \tag{53}$$

After inserting its definition (14), most terms vanish per compatibility of ∇ and what remains is proportional to the Nijenhuis tensor.

(b) $\Longleftrightarrow$ **(c)**: It is straightforward to check that

$$[X\mp iJX,Y\mp iJY]=([X,Y]-[JX,JY])\mp iJ\left([X,Y]-[JX,JY]\right) \tag{54}$$

holds exactly when $\mathrm{Nij}(X,Y)=0$.

(c) $\Longleftrightarrow$ **(d)**: The $(0,2)$-part of $d\alpha$ for any $\alpha\in\Omega^{1,0}_{\mathbb{C}}$ vanishes if and only if

$$0=X(\alpha(Y))-Y(\alpha(X))-\alpha([X,Y])\qquad\forall X,Y\in\Gamma(T''M). \tag{55}$$

The first two terms always vanish while the third term does so for all α exactly when $[X,Y]$ is always antiholomorphic.

(c) $\Longleftrightarrow$ **(e)**: The argument of the last step goes through entirely analogously for differential forms of higher rank.

□

The following important and highly non-trivial theorem asserts that torsion is indeed the only obstruction to integrability of an almost complex manifold:

3.35 Theorem (Newlander-Nirenberg)**.** A manifold equipped with a torsion-free almost complex structure is already integrable.

See e.g. [11] for a proof. In the torsion-free/integrable case, we can conveniently name the two remaining components of the exterior derivative:

3.36 Definition (Splitting of exterior derivative)**.** Let J be an integrable almost complex structure on a manifold M. For all $\alpha \in \Omega_{\mathbb{C}}^{p,q}M$ we define the **Dolbeault operators**

$$d'\alpha := \pi^{p+1,q}(d\alpha), \qquad d''\alpha := \pi^{p,q+1}(d\alpha), \qquad d^c := i(d'' - d'), \tag{56}$$

where $\pi^{p,q}$ is the projection onto $\Omega_{\mathbb{C}}^{p,q}M$.

3.37 (Differential identities)**.** As immediate consequences of this definition and 3.34, we find

$$d = d' + d'', \qquad d' = (d + id^c)/2, \qquad d'' = (d - id^c)/2, \tag{57}$$

$$d'^2 = 0, \qquad d''^2 = 0, \qquad (d^c)^2 = 0, \tag{58}$$

$$d'd'' + d''d' = 0, \qquad dd^c + d^c d = 0, \qquad dd^c = 2id'd''. \tag{59}$$

3.38 (Dolbeault Cohomology)**.** We call a (p,q)-form α d''-closed if $d''\alpha = 0$ and d''-exact if we can find a $(p,q-1)$-form β such that $\alpha = d''\beta$. Since $d''^2 = 0$, d'' is the boundary map of a complex, yielding the groups

$$H_D^{p,q}(M,\mathbb{C}) := \frac{\ker\left(d'' : \Omega_{\mathbb{C}}^{p,q}M \to \Omega_{\mathbb{C}}^{p,q+1}M\right)}{\operatorname{im}\left(d'' : \Omega_{\mathbb{C}}^{p,q-1}M \to \Omega_{\mathbb{C}}^{p,q}M\right)} \tag{60}$$

for $q \geq 0$, where we understand the divisor to be trivial for $q = 0$. This **Dolbeault cohomology** is the complex analogue of the de Rham cohomology and the complex dimensions of $H_D^{p,q}(M,\mathbb{C})$ are called **Hodge numbers** $h^{p,q}(M)$. This cohomology is not a topological invariant since it also depends on the complex structure.

There is a complex version of Poincaré's Lemma which guarantees local triviality analogously to the de Rham case:

3.39 Proposition (Dolbeault-Grothendiek Lemma)**.** Let $\alpha \in \Omega^{p,q}M$ be a (p,q)-form on a simply connected complex manifold. If α is d''-closed, then it is already d''-exact.

For a proof, see e.g. page 25 of [12]. Another consequence of this Lemma is the following:

3.40 Proposition (Local dd^c-Lemma)**.** Let α be a closed and real (1,1)-form on a complex manifold M. We can then locally find a real function f such that $\alpha = dd^c f = 2id'd'' f$.

Proof. By the regular Poincaré Lemma, we can locally find a real β so that $\alpha = d\beta$. Since β is real, we can decompose it into $\beta = \gamma + \bar{\gamma}$, where γ is a $(0,1)$-form. Moreover, since α is a $(1,1)$-form and $\alpha = (d' + d'')(\gamma + \bar{\gamma})$, we find $d''\gamma = d'\bar{\gamma} = 0$. This allows us to use the Dolbeault-Grothendiek Lemma to locally find a complex function g so that $\gamma = d''g$. Putting it all together, we find

$$\alpha = (d' + d'')(d''g + d''\bar{g}) = d'd'' \operatorname{Im} g. \tag{61}$$

Rescaling $\operatorname{Im} g$ yields the real f we are looking for. □

3.5 Complex volume forms as $SL(n/2, \mathbb{C})$-structures

3.41 Definition. We define a **special almost complex structure** on a manifold M of even dimension n as a $SL(n/2, \mathbb{C})$-structure.

3.42 (Alternative description)**.** A special almost complex structure on a manifold M is equivalent to the choice of a complex $n/2$-form τ called **complex volume form** such that[4] for all $p \in M$

$$\dim_{\mathbb{C}} \ker \tau_p = n/2 \qquad \text{and} \qquad \ker \tau_p \cap \overline{\ker \tau_p} = \{0\} \tag{62}$$

or equivalently

$$\tau_p \wedge \overline{\tau_p} \neq 0 \qquad \text{and locally} \qquad \tau_p = \theta_1 \wedge ... \wedge \theta_{n/2} \tag{63}$$

for some $\theta_j \in T^*M$. This holds as usual because the special linear group can be defined as the stabilizer of the standard complex volume form τ_0 over $\mathbb{C}^{n/2}$. Under the almost complex structure discussed in the next remark, the conditions in Eq. (63) simply say that a frame exists where τ takes the form τ_0.

3.43 (Induced almost complex structure)**.** A complex volume form τ naturally induces an almost complex structure J_τ on the manifold given by

$$J_\tau(v_p) := \begin{cases} -iv_p & v_p \in \ker \tau_p \\ iv_p & v_p \in \ker \overline{\tau_p} \end{cases} \tag{64}$$

This immediately yields $\ker \tau = T''M$ and $\overline{\ker \tau} = T'M$. Moreover, τ is of type $(n/2, 0)$, i.e. a section of the holomorphic line bundle $\Lambda^{n/2,0}_{\mathbb{C}} M$ of the induced almost complex structure.

3.44 (Induced real volume form)**.** Since $SL(n/2, \mathbb{C}) \subseteq SL(n, \mathbb{R})$, we also find a natural *real* volume form by setting

$$\mu_\tau = \tau \wedge \overline{\tau}. \tag{65}$$

3.45 Definition. We define a **special complex structure** on a manifold M as an *integrable* $SL(n/2, \mathbb{C})$-structure and refer to the associated τ as the **holomorphic volume form**.

The following proposition motivates the second term:

3.46 Proposition. For a special almost complex structure τ, the following statements are equivalent:

(a) τ is integrable.

(b) τ is torsion-free.

(c) $d\tau = 0$, i.e. τ is a *holomorphic* section of the canonical holomorphic line bundle with respect to the induced almost complex structure.[5]

In particular, the induced almost complex structure is torsion-free/integrable if any of these statements hold.

[4] We define the kernel of a complex form as $\ker \tau_p := \{v \in (T_{\mathbb{C}}M)_p | i_v \tau = 0\}$, where i_v denotes the interior product.

[5] The term holomorphic will become clear in the proof of (c) $\implies$ (a). While we will not go into more detail on this, this is related to the fact that $\Lambda^{n/2,0}_{\mathbb{C}} M$ can be seen as a holomorphic line bundle if the induced almost complex structure is integrable.

Proof.
(a) $\implies$ (b) holds trivially.

(b) $\implies$ (c): Let ∇ be a torsion-free connection that is compatible with the special almost complex structure. We calculate:

$$
\begin{aligned}
d\tau(X_0, ..., X_n) &\overset{\text{Def.}}{=} \sum_{0\le i\le n} (-1)^i \, X_i(\tau(X_0, ..., \hat{X}_i, ..., X_n)) \\
&\quad + \sum_{0\le i<j\le n} (-1)^{i+j} \, \tau([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n) \\
&= \sum_{0\le i\le n} \sum_{j\ne i} (-1)^i \, \tau(X_0, ..., \nabla_{X_i} X_j, ..., \hat{X}_i, ..., X_n) \\
&\quad + \sum_{0\le i<j\le n} (-1)^{i+j} \, \tau(\nabla_{X_i} X_j - \nabla_{X_j} X_i, X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n)
\end{aligned}
$$

In writing out the definition of the exterior derivative in the first equality, we use $\hat{X}_i$ to denote arguments that are skipped over. For the first set of terms in the second equality, we have used the Leibniz rule of ∇ as well as $\nabla\tau = 0$, which by 2.17 is just the compatibility of ∇. For the second set of terms in the second equality, we used the fact that ∇ is torsion-free in writing $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i$. One can then see that the terms cancel each other out by moving the $\nabla_{X_i} X_j$ in the first set of terms to the first argument and rewriting the indices.

Note that the argument presented here is essentially just the general argument that $\nabla\alpha = 0$ implies $d\alpha = 0$ for torsion-free ∇.

(c) $\implies$ (a): We first show that the induced almost complex structure is integrable by considering Eq. (63): Using that $\tau \wedge \theta_j$ and thereby its exterior derivative both vanish, $d\tau = 0$ immediately implies $\forall j : \tau \wedge d\theta_j = 0$. But this just means that the $d\theta_j$ have no component of type $(0, 2)$. Since the θ_j locally span $T'M$, by 3.34(d) this is already sufficient for the almost complex structure to be torsion-free.

Due to the Newlander-Nirenberg theorem 3.35, we can now consider a holomorphic coordinate system $\{z_j\}_{j=1...n/2}$ around any given point of the manifold. The volume form can be expressed as

$$\tau := f(z_1, ..., z_{n/2}) \, dz_1 \wedge ... \wedge dz_{n/2}, \tag{66}$$

where f must be an everywhere non-zero holomorphic complex function since $d\tau = 0$. Taking $g : \mathbb{C}^{n/2} \to \mathbb{C}$ to be a holomorphic function such that $\partial g/\partial z_1 = f$, we perform a holomorphic coordinate change in the first variable according to

$$(z_1, z_2, ...) \mapsto (\tilde{z}_1, z_2, ...) \qquad \text{where} \qquad \tilde{z}_1(x) := g\big(z_1(x), ..., z_{n/2}(x)\big). \tag{67}$$

The Jacobian determinant of this transformation is just $f \neq 0$, so that by the holomorphic version of the local inversion theorem it is indeed a holomorphic coordinate change. In this coordinate system,

$$\tau := d\tilde{z}_1 \wedge dz_2 \wedge ... \wedge dz_{n/2}, \tag{68}$$

so that we have seen that the special almost complex structure is integrable as well. □

3.47 (Volume forms with prescribed almost complex structure)**.** The volume forms that induce a given J clearly stand in one-to-one correspondence to nowhere vanishing sections of the holomorphic line bundle. By the preceding proof, the holomorphic sections are exactly those that correspond to holomorphic, i.e. integrable volume forms. Such global holomorphic sections exist if and only if the canonical holomorphic line bundle is a trivial holomorphic bundle.

3.6 Symplectic manifolds as $Sp(n, \mathbb{R})$-structures

3.48 Definition. An $Sp(n, \mathbb{R})$-structure on a manifold M of even dimension n is called a **symplectic structure** if it is integrable and is otherwise just an **almost symplectic structure**.

3.49 (Alternative description)**.** The symplectic group $Sp(n, \mathbb{R})$ can be defined as the stabilizer of the 2-form ω_0 over $\mathbb{R}^n$ that is represented by the block matrix

$$\omega_0 = \begin{pmatrix} 0 & \mathrm{Id}_{n/2} \\ -\mathrm{Id}_{n/2} & 0 \end{pmatrix}. \tag{69}$$

All non-degenerate 2-forms are pointwise similar to ω_0. This can be seen by inductively constructing a basis similarly to the Gram-Schmidt process, but using the *symplectic* instead of the *orthogonal* complement.

We then see by 2.17 that an almost symplectic structure is equivalent to the choice of a non-degenerate 2-form on M, called the **symplectic form**. Similarly to Riemannian metrics, the symplectic form can also be interpreted as an isomorphism $\varphi_\omega : TM \to T^*M, v_p \mapsto \omega_p(v_p, \cdot)$.

Note that the existence of an almost symplectic structure on a manifold is equivalent to that of almost complex and/or Hermitian structures. This is discussed in more detail in remark 3.57.

3.50 (Induced volume form)**.** Since $Sp(n, \mathbb{R}) \subseteq SL(n, \mathbb{R})$, an almost symplectic structure induces a volume form that can explicitly be written as

$$\mu_\omega = \frac{1}{(n/2)!}\, \omega^{n/2}. \tag{70}$$

3.51 (Intrinsic torsion)**.** Considering 2.18 in the case of an almost symplectic structure ω, we can use φ_ω to identify

$$\mathfrak{sp}(\dim M, \mathbb{R}) \otimes T^*M \simeq (\mathrm{Sym}^2\, T^*M) \otimes T^*M \quad \text{and} \quad TM \otimes \Lambda^2 T^*M \simeq T^*M \otimes \Lambda^2 T^*M. \tag{71}$$

The image of $\tilde{\sigma}$ is then given by tensors that are first symmetrized in the first two indices and then antisymmetrized in the last two so that $\operatorname{coker} \tilde{\sigma} = \Lambda^3 T^*M$ and the associated projection map is just antisymmetrization.

Given any connection ∇ on M, one readily finds

$$d\omega(X, Y, Z) = (\nabla_X \omega)(Y, Z) + \omega(T(X, Y), Z) + \text{cyclic permutations} \tag{72}$$

by replacing Lie brackets using the definition of torsion and simplifying. For compatible connections, the first set of terms vanish and what remains is exactly the intrinsic torsion under the identification introduced above. An almost symplectic structure is therefore torsion-free if and only if the form ω is closed.

A theorem similar to the complex case holds:

3.52 Theorem (Darboux [13])**.** A torsion-free almost symplectic structure is integrable.

In other words, a symplectic structure is just a closed non-degenerate 2-form ω. For a proof of Darboux's Theorem based on a trick Moser used for his Theorem 3.9, see e.g. [14].

3.53 (Poisson bracket and Hamiltonian flow)**.** Symplectic manifolds are motivated as a generalization of the phase space of Hamiltonian mechanics, which is just the cotangent bundle T^*Q of a manifold Q that represents the position space of a system together with the negative exterior derivative of the tautological one-form as the symplectic form.

The generalization to symplectic manifolds M is constructed in such a way as to keep essential tools intact: Given an energy function $H \in C^\infty(M)$, the symplectic form induces the **Hamiltonian vector field** X_H via

$$dH = \omega(X_H, \cdot), \tag{73}$$

so that the integral curves conserve ω and H. We can also define the bilinear **Poisson bracket**

$$\{f, g\} := \omega(X_f, X_g) = X_f(g) \qquad \forall f, g \in C^\infty(M), \tag{74}$$

which is skew-symmetric and fulfills Leibniz's rule and the Jacobi identity.

3.7 Hermitian and Kähler manifolds as $U(n/2)$-structures

3.54 Definition. An **almost Hermitian structure** on a manifold M of even dimension n is a $U(n/2)$-structure.

The unitary group is naturally embedded into a number of larger groups since

$$U(n/2) = O(n) \cap Sp(n, \mathbb{R}) \cap GL(n/2, \mathbb{C}). \tag{75}$$

For this reason, a $U(n/2)$-structure on a manifold M automatically induces a Riemannian, almost symplectic and almost complex structure. The fact that the intersection of any two of these three groups recovers the unitary group is reflected in the equivalent ways to explicitly characterize the unitary structure:

3.55 Proposition. An almost Hermitian structure on a manifold M induces an almost complex structure J, a Riemannian metric g and an almost symplectic structure ω that are compatible in the sense that the following equivalent equations hold:

$$g(X, Y) = \omega(X, JY) \tag{76}$$

$$\omega(X, Y) = g(JX, Y) \tag{77}$$

$$J(X) = \varphi_g^{-1} \varphi_\omega(X) \tag{78}$$

Contrarily, a manifold M equipped with only two structures out of J, g and ω is already almost Hermitian if and only if the uniquely determined third object defined by these equations fulfills the respective axioms of the missing structure.

Given J and a Riemannian metric g, this is the case exactly when g is J-invariant. Given J and an almost symplectic form ω instead, it is the case when ω is both J-invariant and positive in the sense of remark 3.29.

Proof. Eqs. (76)-(78) follow directly from the corresponding equations that hold for the tensors over $\mathbb{R}^n$ that J, g and ω are modeled on.

Now let J, g and ω represent any almost complex, Riemannian and almost symplectic structures that are compatible in the sense that these equations hold. In every point of M, we can choose a frame of the form $(e_1, ..., e_{n/2}, Je_1, ..., Je_{n/2})$. We can use the metric g to Gram-Schmidt orthonormalize $(e_1, ..., e_{n/2}) \mapsto (\tilde{e}_1, ..., \tilde{e}_{n/2})$, which extends to an orthonormal frame $(\tilde{e}_1, ..., \tilde{e}_{n/2}, J\tilde{e}_1, ..., J\tilde{e}_{n/2})$ since g is J-invariant. In this frame, J, g and ω are given by the tensors they are modeled on because Eq. (77) determines ω uniquely. This means that the intersection P of the bundles corresponding to each of the structures is everywhere non-empty. Since the unitary group can be defined as the linear transformations that simultaneously preserve the model tensors, any two frames within a fiber of P are related by exactly one element of $U(n/2)$, turning P into an almost Hermitian structure.

Finally, note that for a fixed J, ω and g have to be associated forms in the sense of remark 3.29. The statements of that remark trivially yield the conditions on ω or g that suffice to construct an almost Hermitian structure by defining the remaining object by one of Eqs. (76)-(78). □

Reexpressing a J-invariant Riemannian metric from a complex perspective yields yet another equivalent way to describe an almost Hermitian structure:

3.56 (Hermitian metrics)**.** Given an almost Hermitian structure (J, g, ω) on a manifold M, we define the **Hermitian metric** h by

$$h(X,Y) := g\left(\frac{X - iJX}{2}, \frac{Y + iJY}{2}\right) \qquad \forall X, Y \in (T_H M)_{x \in M}, \tag{79}$$

where we have $\mathbb{C}$-linearly continued the metric g to the complexified tangent bundle. The Hermitian metric is a symmetric sesquilinear form on the holomorphic vector bundle $T_H M$.

It is straightforward to check that it can be expressed through the metric and almost symplectic form:

$$h = (g - i\omega)/2. \tag{80}$$

Conversely, taking the real part of any such symmetric sesquilinear form and doubling it recovers the J-invariant Riemannian metric, which according to the previous proposition suffices to recover the whole almost Hermitian structure.

3.57 (Existence)**.** An almost Hermitian structure exist on a manifold exactly when an almost complex or, equivalently, almost symplectic structure exists: Given only an almost complex structure J, we can pick any metric g and turn it J-invariant by setting

$$\tilde{g}(X,Y) := g(X,Y) + g(JX, JY) \qquad \forall X, Y \in T_p M, p \in M, \tag{81}$$

which yields an almost Hermitian structure by 3.55.

Conversely, given an almost symplectic structure ω, we can again pick any metric g and define a section $A := \varphi_g^{-1} \circ \varphi_\omega$ of the endomorphism bundle. While A is not in general an almost complex structure, the orthogonal part J of its polar decomposition is. One can show that ω is both J-invariant and positive definite with respect to J, so that by 3.55 we obtain an almost Hermitian structure again. The details of this argument are laid out in Proposition 12.3 of [15]. Note that the induced metric is in general different from the g we started out with.

Clearly, for a manifold to admit an almost Hermitian structure it must be orientable and of even dimension. This is not always sufficient: Among the spheres, it turns out that only S^2 and S^6 can be equipped with one. One concrete example of a topological obstruction is that all odd-dimensional Stiefel-Whitney classes have to vanish if the manifold is compact (e.g. page 171 of [16]).

3.58 (Real volume form)**.** The almost Hermitian structure of course also induces a $SL(n, \mathbb{R})$ structure, i.e. a real volume form μ. It can be characterized with either the oriented Riemannian or almost symplectic structures as in 3.16 and 3.50. This also implies J-invariance of the volume form.

3.59 (Local description in holomorphic coordinates)**.** Given holomorphic coordinates $\{z_a = x_a + iy_a\}_{a=1,\ldots,n/2}$, symmetry and J-invariance imply that the metric can be written as

$$g \;=\; g^1_{ab}\left(dx^a \otimes dx^b + dy^a \otimes dy^b\right) \;+\; g^2_{ab}\left(dx^a \otimes dy^b - dy^a \otimes dx^b\right), \tag{82}$$

where g^1 and g^2 are symmetric and antisymmetric real matrices, respectively. The $\mathbb{C}$-linear continuation of g can then be rewritten to

$$g \;=\; g_{a\bar{b}}\left(dz^a \otimes d\bar{z}^b + d\bar{z}^b \otimes dz^a\right), \tag{83}$$

where $g_{a\bar{b}} = g_{\bar{b}a} = (g^1_{ab} + ig^2_{ab})/2$ are Hermitian matrices. The almost symplectic form can be represented as

$$\omega \;=\; 2ig_{a\bar{b}}\left(dz^a \wedge d\bar{z}^b\right) \;=\; ig_{a\bar{b}}\left(dz^a \otimes d\bar{z}^b - d\bar{z}^b \otimes dz^a\right), \tag{84}$$

which by 3.50 yields the volume form

$$\mu = (2i)^{n/2} \det(g_{a\bar{b}})\, dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge ... \wedge dz^{n/2} \wedge d\bar{z}^{n/2}. \tag{85}$$

The Hermitian metric on the holomorphic tangent bundles is given by

$$h \;=\; g_{a\bar{b}}\, dz^a \otimes d\bar{z}^b, \tag{86}$$

where dz^a and $d\bar{z}^b$ are understood to act on vectors in $T_H M$ by first applying the respective linear and skewlinear isomorphisms from 3.26.

Hermitian manifolds and Dolbeault-Hodge Theory

3.60 Definition. A **Hermitian structure** is an almost Hermitian structure such that the induced almost complex structure is, in fact, complex.

Our main motivation for considering Hermitian structures here is that they have both well-defined Dolbeault operators and an oriented Riemannian structure that gives a $\mathbb{C}$-linearly continued Hodge star operator. Together, this allows us to extensively mimic Hodge theory, but considering d'' instead of d:

3.61 (Dolbeault-Hodge Codifferential and Laplacian)**.** We define the **codifferential** $d''^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$ as

$$d''^* := - * \circ\, d' \circ * \tag{87}$$

for $k > 0$ and $d''^* f = 0$ for functions $f \in \Omega^0(M)$. The sign appears simpler than in the Riemannian case just because we know that the dimension of a Hermitian manifold is even. We use d' on the right hand side so that d''^* is the formal adjoint to d'' with respect to the *Hermitian* L^2-inner product $< \alpha, \beta >_{L^2,H} := < \alpha, \bar{\beta} >_{L^2}$. Again, $(d''^*)^2 = 0$ and we say α is d''**-coclosed** if $d''^*\alpha = 0$ and d''**-coexact** if there exists a β such that $\alpha = d''^*\beta$.

The **Dolbeault-Hodge-Laplacian** $\Delta_{d''} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$ is defined as

$$\Delta_{d''} = d''d''^* + d''^*d'' = (d'' + d''^*)^2. \tag{88}$$

A k-form α is called d''**-harmonic** if $\Delta_{d''}\,\alpha = 0$, which again is equivalent to being both d''-closed and d''-coclosed. We write $\mathcal{H}^{p,q}_{d''}(M) := \ker\big(\Delta_{d''} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)\big)$ for the space of such forms.

Note that we could similarly have considered codifferentials and Laplacians of d' or d^c, but this is not necessary for our discussion. We find a completely analogous decomposition theorem and corollaries:

3.62 Theorem (Dolbeault-Hodge Decomposition Theorem)**.** Let M be a compact Hermitian manifold. The space of (p, q)-forms decomposes into

$$\Omega^{p,q}_{\mathbb{C}}(M) = \mathcal{H}^{p,q}_{d''} \oplus d''\Omega^{p,q-1}(M) \oplus d''^*\Omega^{p,q+1}(M), \tag{89}$$

and the three constituent spaces are orthogonal with respect to the Hermitian L^2-inner product.

The elaborate proof of the direct sum decomposition can be found starting on page 84 of [12]. The proofs of the following two corollaries work precisely as those of 3.20 and 3.21 in the Riemannian case.

3.63 Corollary. Let M be a compact Hermitian n-manifold. Every element of the (p, q)-th Dolbeault cohomology group contains exactly one harmonic (p, q)-form so that $H^{p,q}_D(M)$ is naturally isomorphic to $\mathcal{H}^{p,q}_{d''}(M)$.

3.64 Corollary (Serre duality)**.** Let M be a compact Hermitian n-manifold. The spaces $\mathcal{H}^{p,q}_{d''}(M)$ and $\mathcal{H}^{n/2-p,n/2-q}_{d''}(M)$ are naturally isomorphic and the Hodge numbers satisfy $h^{p,q}(M) = h^{n/2-p,n/2-q}(M)$.

Here, the ($\mathbb{C}$-skewlinear) isomorphism is given by the composition of the Hodge star and complex conjugation.

We cannot yet make statements about the compatibility between the Riemannian and Dolbeault Hodge theory and the usual de Rham cohomology. It will turn out that the missing ingredient for this is the topic of the next section, namely a Kähler structure.

Kähler manifolds

Erich Kähler opened up the investigation into a class of particularly convenient Hermitian structures in 1933 [17]:

3.65 Definition. A **Kähler structure** is a torsion-free $U(n/2)$-structure.

Clearly, a Kähler structure automatically yields a complex and a symplectic structure, since any torsion-free $U(n/2)$-connection is also compatible with the induced $Gl(n/2, \mathbb{C})$ and $Sp(n, \mathbb{R})$-structures. Conversely, the following result allows us to characterize Kähler structures:

3.66 Proposition. Let M be a manifold equipped with an almost Hermitian structure expressed through (g, ω, J) and let ∇^g be the Levi-Civita connection of g. The following statements are equivalent:

(a) The almost Hermitian structure is Kähler, i.e. torsion-free.

(b) The induced almost complex and almost symplectic structures are both torsion-free or, equivalently, integrable[6].

(c) The almost complex structure J is ∇^g-parallel, i.e. $\nabla^g J = 0$.

(d) The almost symplectic structure is ∇^g-parallel, i.e. $\nabla^g \omega = 0$.

(e) Around each point of M, there exist holomorphic charts under which one (equivalently all) of the Riemannian metric, Hermitian metric and almost symplectic form osculate to second order at that point:

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} + O(|z|^2). \tag{90}$$

Proof.
(a) $\implies$ **(b)** immediately by definition.

(b) $\implies$ **(c)** follows directly from the formula

$$2\, g((\nabla^g_X J)Y, Z) = d\omega(X, JY, JZ) - d\omega(X, Y, Z) - g(\mathrm{Nij}(Y, Z), JX) \tag{91}$$

for all vector fields X, Y and Z since we identified the intrinsic torsions with $d\omega$ and Nij in 3.51 and 3.34, respectively. To see that Eq. (91) holds, use $(\nabla^g_X J)Y = \nabla^g_X (JY) - J\nabla^g_X Y$ and Eq. (25) to rewrite the left-hand side. Using Eqs. (76)-(78) and the definitions of the exterior derivative and Nijenhuis tensor, the remaining terms can be seen to equal the right-hand side in a direct but lengthy calculation.

[6] This can of course be characterized in various ways, see 3.34 and 3.51.

(c) $\iff$ (d): If either the complex or symplectic structure is ∇^g-parallel then the other must be, too, due to Eqs. (76)-(78) and $\nabla^g g = 0$.

(c/d) $\implies$ (a): (c) and (d) mean that the Levi-Civita connection ∇^g is compatible with both induced structures so that the connection form Ω^g takes values in the Lie algebras of both of the respective groups. But the intersection of these is just the Lie algebra of the Hermitian structure, so that the Levi-Civita connection is compatible with it as well. Since ∇^g is always torsion-free, we have shown that the Hermitian structure must lack intrinsic torsion.

(b) $\implies$ (e): We can choose a holomorphic coordinate system in every point of M since J is integrable. After expressing ω in it, we can construct a holomorphic coordinate change using the local inversion theorem and $d\omega = 0$ in such a way that it exactly cancels the linear terms of $g_{a\bar{b}}$. This argument is laid out in detail on page 29 of [18].

(e) $\implies$ (b): Since M can be covered with holomorphic charts, the almost complex structure is integrable. Either all or none of the mentioned structures osculate due to the local expressions stated in 3.59. The exterior derivative of the almost symplectic structure evaluates to

$$d\omega = \left(\frac{\partial g_{a\bar{b}}}{\partial z^c} dz^c + \frac{\partial g_{a\bar{b}}}{\partial \bar{z}^c} d\bar{z}^c \right) \wedge dz^a \wedge d\bar{z}^b, \tag{92}$$

which must vanish in every point by considering it in the coordinates where $g_{a\bar{b}}$ osculates to second order. This implies that the almost symplectic structure is torsion-free. □

Note that condition (e) is particularly useful in that it allows us to conclude that every identity that depends only on a Kähler metric and its first derivatives holds exactly if it holds for the flat metric in $\mathbb{C}^{n/2}$!

3.67 (Examples)**.** A typical example of a Kähler manifold, besides the trivial $\mathbb{C}^m$, is the complex projective space $\mathbb{C}P^m$ equipped with the **Fubini-Study metric**. The usual charts

$$\Psi_j(w_1, ..., w_m) := [w_1, ..., w_{i-1}, 1, w_i, ..., w_m] \tag{93}$$

that cover the open subsets containing those rays that are not parallel to the jth axis are readily checked to have holomorphic transition functions and thereby yield a complex structure. Moreover, we can consistently define a real $(1,1)$-form ω that can for any j be represented as

$$\omega = dd^c \log(1 + |\Psi_j^{-1}|^2) \tag{94}$$

on the patch covered by Ψ_j. One can check that this form is in fact positive and so defines a Kähler structure.

On a complex submanifold of $\mathbb{C}P^m$, i.e. one where the restriction of J is a complex structure itself, the Kähler structure naturally restricts to a Kähler structure as well. This connects Kähler geometry to algebraic geometry: By this reasoning, all projective complex varieties without singular points are Kähler manifolds, yielding a large class of interesting examples. For a closer discussion of this, see e.g. [12].

3.68 (Integrability)**.** One can calculate that the curvature of the Fubini-study metric is strictly positive. This means that, contrary to the complex and symplectic cases, Kähler structures are not necessarily integrable despite being torsion-free already. Otherwise, the induced Riemannian structure would have to be integrable too and thereby flat.

Curvature of Kähler manifolds

3.69 (Riemann Curvature)**.** On a Kähler manifold, the Levi-Civita connection preserves the complex structure. This immediately gives us one additional symmetry of the curvature tensor:

$$R(X \wedge Y)J = JR(X \wedge Y) \qquad \forall X, Y \in T_pM, p \in M.. \tag{95}$$

Combined with its other symmetries, this also implies J-invariance of the Ricci tensor. This allows us to define the **Ricci form** as the associated form to the Ricci tensor in the sense of 3.29:

$$\rho(X, Y) := \mathrm{Ric}(JX, Y) \qquad \forall X, Y \in T_pM, p \in M. \tag{96}$$

3.70 Proposition. Let M be a Kähler manifold. The Ricci form ρ satisfies

$$\rho = i\,\mathrm{Tr}^{\mathbb{C}}(R), \tag{97}$$

where R is the Riemannian curvature tensor. Moreover, the Ricci form is i times the curvature form on the canonical holomorphic line bundle $\Lambda_{\mathbb{C}}^{n/2,0}M$ that is induced by the Levi-Civita connection.

Proof. For any two vectors X and Y at a point, we calculate in local holomorphic coordinates:

$$\begin{aligned}
i\,\mathrm{Tr}^{\mathbb{C}}(R)(X,Y) &= i\,dz^j\Big(R(X \wedge Y)\partial z_j\Big) \\
&= (i\,dx^j - dy^j)\left(R(X \wedge Y)\left(\frac{\partial x_j - i\partial y_j}{2}\right)\right) \\
&= \frac{1}{2}\Big(dx^j(R(X \wedge Y)\partial y_j) - dy^j(R(X \wedge Y)\partial x_j)\Big) \\
&\quad + \frac{i}{2}\Big(dx^j(R(X \wedge Y)\partial x_j) + dy^j(R(X \wedge Y)\partial y_j)\Big) \\
&= \frac{1}{2}\Big(- dx^j(R(Y \wedge \partial y_j)X) - dx^j(R(\partial y_j \wedge X)Y) \\
&\qquad + dy^j(R(Y \wedge \partial x_j)X) + dy^j(R(\partial x_j \wedge X)Y)\Big) \\
&= \frac{1}{2}\Big(dx^j(R(\partial x_j \wedge JX)Y) + dy^j(R(\partial y_j \wedge JX)Y) \\
&\qquad - dx^j(R(\partial x_j \wedge JY)X) - dy^j(R(\partial y_j \wedge JY)X)\Big) \\
&= \frac{1}{2}\Big(\mathrm{Ric}(JX,Y) - Ric(JY,X)\Big) \;=\; \rho(X,Y)
\end{aligned}$$

For the fourth equality, we applied the algebraic Bianchi identity to the real part and the imaginary terms cancel by an antisymmetry of R. The fifth equality follows by swapping symmetries and J-invariance.

The top exterior power of the holomorphic cotangent bundle arises through the complex determinant as an associated bundle to the frame bundle. Since its pushforward is $\det_* = \mathrm{Tr}$, the previous calculation also identifies the Ricci form as the curvature form on this associated bundle, up to the factor i. □

3.71 Proposition. Let M be a Kähler manifold. The Ricci form ρ can be written as

$$\rho = -\frac{1}{2} dd^c \log\det\left(g_{c\bar{d}}\right). \tag{98}$$

In particular, we see that ρ is closed, and comparing it with Eq. 85 shows that it only depends on the complex structure and volume form instead of the whole Kähler structure.

Proof. By explicit calculation of Christoffel symbols, one can verify the local expression

$$R_{a\bar{b}} = -\partial z_a \partial \bar{z}_b \left(\log \det(g_{c\bar{d}})\right). \tag{99}$$

for the complexified Ricci tensor. This calculation can be found in Remark 6.2 of [18]. We do not reproduce this here and instead follow [19] in using the preceding proposition to derive Eq. (98) in a more structural manner:

As in 2.9, we choose a local section σ to represent the (Levi-Civita) covariant derivative as $\eta(X)\sigma := \nabla_X \sigma$ and the curvature as $R(X \wedge Y)\sigma = d\eta(X,Y)\sigma$. We will show $\eta = d' \log \det(g_{a\bar{b}})$, because this yields Eq. (98) after using $dd^c = -2id'd''$ and Proposition 3.70.

The Hermitian scalar product on the tangent bundle described in 3.56 induces a Hermitian scalar product $h(\alpha, \beta) := g(\alpha, \bar{\beta})$ on the canonical holomorphic line bundle. Compatibility of the Levi-Civita connection means

$$X(h(\sigma,\sigma)) = h(\nabla_X \sigma, \sigma) + h(\sigma, \nabla_X \sigma) = (\eta + \bar{\eta})(X) \cdot h(\sigma, \sigma), \tag{100}$$

or equivalently

$$\eta + \bar{\eta} = d \log h(\sigma, \sigma). \tag{101}$$

On a Kähler manifold, the Levi-Civita connection is also compatible with the complex structure. If we therefore choose a holomorphic local section for σ, as in 3.46, then $\nabla_X \sigma$ has to vanish if $X \in T''M$. This implies that η is of type $(1,0)$. With Eq. (101), we conclude $\eta = d' \log h(\sigma, \sigma)$. In a given local holomorphic coordinate chart, $\sigma := dz^1 \wedge \ldots \wedge dz^{n/2}$ is a holomorphic section with $h(\sigma, \sigma) = \det(g_{a\bar{b}})$ by Eq. (24) so that indeed $\eta = d' \log \det(g_{a\bar{b}})$. ☐

As a closed form, ρ represents an element of de Rham cohomology. And not just any element:

3.72 Proposition. For a Kähler manifold M with Ricci form ρ and first real Chern class $c_1(M) \in H^2(M, \mathbb{R})$,

$$[\rho] = 2\pi\, c_1(M) \tag{102}$$

holds.

Proof. The first real Chern class can be defined through the representative $\underline{c}_1 := -\frac{1}{2\pi i} \operatorname{Tr}^{\mathbb{C}}(R)$, so that Proposition 3.70 immediately gives this result. ☐

3.73. We can define **Kähler-Einstein** manifolds to be a manifold equipped with a Kähler structure so that the induced Riemannian structure is Einstein, i.e. the Ricci tensor is proportional to the metric. Since both of them arise as associated forms, this is clearly also equivalent to the Ricci form being proportional to the symplectic form. We will also see later on that the special case of Ricci-flat Kähler manifolds is equivalent to being able to locally find a compatible complex volume form.

Kähler manifolds and Dolbeault-Hodge Theory

3.74. One of the central characteristics of Kähler manifolds with far-reaching consequences is that our different definitions of Laplacians essentially coincide:

$$\Delta_d = 2\Delta_{d''}. \tag{103}$$

We will give an overview of the principles behind this but refer to chapter 0.7 of [12] for the full calculation. The operator $L(\alpha) := \alpha \wedge \omega$ on k-forms fulfills the so-called *Kähler identities*, which give the possible commutators of L, the Dolbeault operators and their adjoints. These identities all trivially follow from $[L, d^*] = d^c$, which can be checked on $\mathbb{C}^n$ in a somewhat notationally intensive calculation. But because it contains only first derivatives of the metric, it can be brought over to any Kähler manifold: By 3.66(e), every point has a coordinate chart where the metric looks like the standard metric up to corrections of second order. Given the Kähler identities and in particular that d' and d''^* anticommute, deriving Eq. (103) is just a matter of some algebraic reshuffling.

Eq. (103) has important consequences:

3.75 Corollary. If M is a compact Kähler manifold, Dolbeault cohomology is a refinement of complex de Rham cohomology:

$$H^k_{dR}(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_D(M,\mathbb{C}) \qquad \text{and} \qquad \mathcal{H}^k_d(M,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{d''}(M). \tag{104}$$

Proof. Clearly, $\bigoplus_{p+q=k} \mathcal{H}^{p,q}_{d''}(M) \subseteq \mathcal{H}^k_d(M,\mathbb{C})$ due to Eq. (103). Moreover, since $\Delta_{d''}$ preserves form types by construction, the same must hold for Δ_d. In particular, it commutes with the type decomposition of any $\alpha \in \mathcal{H}^k_d(M,\mathbb{C})$, so that we also have $\bigoplus_{p+q=k} \mathcal{H}^{p,q}_{d''}(M) \supseteq \mathcal{H}^k_d(M,\mathbb{C})$. The decomposition of de Rham cohomology groups then follows by identifying them with the spaces of harmonic forms via 3.20 and 3.63. □

3.76 Corollary. If M is a compact Kähler manifold, complex conjugation is a natural isomorphism between the respective harmonic spaces:

$$\mathcal{H}^{p,q}_{d''}(M) \cong \mathcal{H}^{q,p}_{d''}(M) \qquad \text{so that} \qquad H^{p,q}_D(M,\mathbb{C}) \cong H^{q,p}_D(M,\mathbb{C}). \tag{105}$$

Proof. Δ_d is a real operator, so by Eq. (103) $\Delta_{d''}$ is, too. It therefore commutes with complex conjugation, making it preserve harmonicity of forms. □

3.77 Corollary. The following relations of Hodge and Betti numbers hold on every compact Kähler n-manifold M:

$$b_k(M) = \sum_{p+q=k} h^{p,q}(M), \qquad h^{p,q}(M) = h^{q,p}(M), \qquad h^{l,l}(M) \geq 1, \tag{106}$$

where $0 \leq l \leq \frac{n}{2}$.

Proof. The preceding corollaries imply the first two statements by definition. We have already seen that the closed symplectic form $\omega^{n/2}$ is proportional to the non-zero volume form, so $\alpha := \omega^l$ is a closed non-zero (l,l)-form for all $0 \leq l \leq \frac{n}{2}$. Just like ω, α and $*\alpha$ are parallel with respect to the Levi-Civita connection. As we have seen in the proof of 3.46 (b) $\Longrightarrow$ (c), this implies $d\alpha = d*\alpha = 0$, so that α is harmonic. Hodge's Theorem then gives the remaining statement. □

In particular, this means that the Betti numbers b_k of a compact Kähler manifold with even k are non-zero and the ones with odd k are even. We can also generalize the local dd^c-Lemma 3.40:

3.78 Corollary (Global dd^c-Lemma)**.** Let $\alpha \in \Omega^{1,1}_{\mathbb{C}}(M)$ be a real exact $(1,1)$-form on a compact Kähler manifold. Then we can find a function $f \in C^\infty(M)$ such that

$$\alpha = dd^c f = 2id'd''f. \tag{107}$$

This is unique up to a constant on every connected component of M.

Proof. Because α is d-exact, it is also d' and d''-closed. Per the Hodge Decomposition with respect to Δ_d, α is orthogonal to the space of harmonic forms. It does not matter with respect to which operator one defines harmonic here due to Eq. (103).

Therefore, the Hodge Decomposition with respect to $\Delta_{d'}$ is $\alpha = d'\beta + d'^*\gamma$: Because α is d'-closed, $0 = d'd'^*\gamma$ and in particular $0 =< \gamma, d'd'^*\gamma >_{L^2} = ||d'^*\gamma||_{L^2}$ so that $\alpha = d'\beta$.

Next up is the Hodge Decomposition with respect to $\Delta_{d''}$ of $\beta = \epsilon^H + d''\epsilon_1 + d''^*\epsilon_2$, where ϵ^H is harmonic. Applying d' gives $\alpha = d'd''\epsilon_1 - d''^*d'\epsilon_2$, where we have used that the Kähler identities imply that d' and d''^* anti-commute. Because α is d''-closed, $0 = d''d''^*d'\epsilon_2$ and in particular $0 =< d'\epsilon_2, d''d''^*d'\epsilon_2 >_{L^2} = ||d''^*d'\epsilon_2||_{L^2}$. What remains is just $\alpha = d'd''\epsilon_1$.

Rescaling ϵ_1 yields the function f we are looking for. Since d and d^c are real operators, it can be chosen real as well. It is unique up to a constant on every connected component: For two $f_{1/2}$ that fulfill the dd^c-Lemma, $d'd''(f_1 - f_2) = 0$ holds. One can check e.g. in local coordinates that this suffices to make the function $f_1 - f_2$ holomorphic, which by the maximum principle fixes it to a constant since M is compact and connected. □

3.79 (Reformulation of the global dd^c-Lemma)**.** In particular we find for any two real $\alpha, \alpha' \in \Omega^{1,1}M$ in the same cohomology class that the dd^c-Lemma applies to their (exact) difference so that there is an $f \in C^\infty(M)$ with

$$\alpha' - \alpha = dd^c f, \tag{108}$$

which is unique up to a constant on every connected component of M.

3.80 (Kähler potential)**.** The local dd^c-Lemma applies to the symplectic form itself so that we can locally define the **Kähler potential** $\phi \in C^\infty(M)$ as

$$\omega = dd^c\phi. \tag{109}$$

The observation is essentially what motivated Kähler to consider these structures. We can locally describe all of the Kähler structure using ϕ by evaluating this equation in complex coordinates and comparing with 3.59, which yields

$$g_{a\bar{b}} = \frac{\partial^2\phi}{\partial z^a \partial \bar{z}^b}. \tag{110}$$

Note that global Kähler potentials do not in general exist. Indeed, on a *compact* Kähler manifold, it never does: The volume form is proportional to a power of the exact symplectic form. Since the integral is invariant under addition of exact forms, this would imply vanishing volume.

3.8 $SU(n/2)$-structures and Calabi-Yau manifolds

3.81 Definition. A **special almost Hermitian structure** on a manifold M of even dimension n is a $SU(n/2)$-structure. It is a **special Kähler structure** if it is also torsion-free.

Clearly, a special almost Hermitian structure induces an almost Hermitian and a special almost complex structure. Since

$$SU(n/2) = U(n/2) \cap SL(n/2, \mathbb{C}) = O(n) \cap SL(n/2, \mathbb{C}) = Sp(n, \mathbb{R}) \cap SL(n/2, \mathbb{C}), \tag{111}$$

we can describe it as a complex volume form that is, in an appropriate sense, compatible with one of the other structures:

3.82 (Alternative description)**.** A special almost Hermitian structure can equivalently be described through an almost Hermitian structure $(g, \omega, J_{g/\omega})$ together with a special almost complex structure τ such the induced almost complex structure and real volume form match up:

$$J_{g/\omega} = J_\tau \qquad \text{and} \qquad \mu_g = \mu_\tau. \tag{112}$$

This is just the usual compatibility condition that there exist frames in which the structures simultaneously take the standard form. To see this in any point $p \in M$, pick a frame where g, ω and $J_{g/w}$ are given by the objects they are modeled on. $J_{g/\omega} = J_\tau$ then tells us by 3.47 that τ equals some constant c times the standard complex volume form in that point p. But $\mu_g = \mu_\tau = \tau \wedge \bar{\tau}$ then implies $|c|^2 = 1$, so rotating one basis vector of the frame by multiplying with c brings τ into standard form while leaving the other tensors invariant.

Using 3.55, this can of course be restated as compatibility of just a Riemannian metric or just an almost symplectic form with the complex volume form and its induced almost complex structure.

3.83 (Intrinsic torsion)**.** The Levi-Civita connection is the only connection compatible with the induced Riemannian structure. For this reason, a special almost Hermitian structure is special Kähler/torsion-free if and only if the induced almost Hermitian structure is Kähler and the Levi-Civita covariant derivative of the complex volume form vanishes. For the first of these, various characterizations can of course be found in 3.66.

3.84 (Existence)**.** A Kähler structure on a manifold can *locally* be reduced to a special Kähler manifold exactly if it is Ricci-flat: 3.70 says that the Ricci form is essentially the Levi-Civita curvature of the holomorphic line bundle. If it is flat, then a parallel section exists locally by 2.9. This is just a compatible local special complex structure according to 3.47 and 3.82. Globally, 3.47 means that such a reduction is possible exactly if the holomorpic line bundle is trivial.

3.85 Definition. A **Calabi-Yau manifold** is a compact and Ricci-flat Kähler manifold.

Given the previous remarks, this is just a compact Kähler structure that can at least locally be equipped with a compatible holomorphic volume form. Note that numerous inequivalent definitions of Calabi-Yau manifolds can be found in the literature. The Calabi-Yau Theorem, which is the topic of the next chapter, motivates this definition since it yields examples of Calabi-Yau manifolds.

3.86 (Physical applications and conjectures)**.** It is well known that consistently formulating realistic string theories requires more dimensions than the usual four of spacetime. To be compatible with experimental observation, the surplus dimension need to be *compactified*, i.e. need to take the form of a compact manifold with a diameter so small as to be unobservable at accessible length scales. Usually, string theories incorporating supersymmetry are considered since they allow for fermionic degrees of freedom in line with the known particles. It turns out that such superstring theories require Calabi-Yau manifolds for compactification (for more see e.g. [20]). Since their exact shape largely determines physical laws at larger length scales, string theorists have a great interest in constructing examples of these manifolds.

Physically motivated correspondences with conformal field theories have led to a number of interesting mathematical conjectures. For example, *mirror symmetry* has received a large amount of interest and has been successfully formalized in some contexts.

Chapter 6.10 of [4] gives a list of references for further reading on Calabi-Yau manifolds.

3.9 Beyond complex structures

There is a number of G-structures that come up as natural extensions of those that we have discussed up to now. We want to give an extremely brief overview:

3.87 (Hypercomplex structures)**.** As we moved from real to complex manifolds, we can move on to quaternions. A **almost hypercomplex structure** is a $GL(n/4, \mathbb{H})$-structure on a manifold whose dimension n is a multiple of four. It can naturally be understood as three almost complex structures I, J and K so that

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}, \tag{113}$$

which makes the tangent spaces isomorphic to $\mathbb{H}^{n/4}$. The structure is **hypercomplex** if it is torsion-free or, equivalently, if I, J and K are. As opposed to the complex case, this does not suffice to imply integrability. The torsion-free connection on a hypercomplex manifold is uniquely determined and called the **Obata connection**.

3.88 (Hyperkähler structures)**.** The subgroup of the general linear quaternionic group that preserves the standard Hermitian form on $\mathbb{H}^{n/4}$ regarded as $\mathbb{R}^n$ is the compact symplectic group

$$Sp(n/4) = Sp(n/2, \mathbb{C}) \cap U(n/2). \tag{114}$$

This group corresponds to an **almost Hyperkähler structure**. It can alternatively be characterized as three almost Hermitian structures that share the same Riemannian metric and whose almost complex structures assemble into an almost hypercomplex structure. It is called **Hyperkähler** if it is torsion-free, which is exactly the case if it consists of three Kähler structures. Moreover, it is always Ricci-flat.

3.89 (Quaternionic and quaternion-Kähler structures)**.** There is a slightly larger subgroup of $GL(n, \mathbb{R})$ that can be associated with quaternions, namely $GL(n/4, \mathbb{H})GL(1, \mathbb{H})$. The corresponding structures are called **(almost) quaternionic**, and can be analogously reduced to **(almost) quaternionic-Kähler** structures with the group $Sp(n/4)Sp(1)$. While the latter are not always Ricci-flat, they are still Einstein.

The only groups appearing in Berger's classification of Riemannian holonomy groups that are still missing are the seven and eight dimensional compact G_2 and $\mathrm{Spin}(7)$, respectively.

4 The Calabi-Yau Theorem

We have previously seen in 3.72 that the Ricci form ρ on a Kähler manifold is a real and closed $(1,1)$-form that represents 2π times the first real Chern class $c_1(M)$. Conversely, the celebrated Calabi-Yau Theorem holds:

4.1 Theorem (Calabi-Yau)**.** Let M be a compact Kähler manifold with symplectic form ω. Given any real $(1,1)$-form ρ' that is cohomologous to its Ricci form, there exists a unique Kähler structure on M with a symplectic form ω' that is cohomologous to ω and whose Ricci form is given by ρ'.

In other words, we can realize *any* real $(1,1)$-form that represents $2\pi\, c_1(M)$ as the Ricci form of a Kähler structure in the same cohomology class. Eugenio Calabi conjectured Theorem 4.1 in 1954 [21, 22] and proved the uniqueness of the adapted Kähler structure. The existence theorem on the other hand was completed only in 1976 by Shing-Tung Yau [23, 24].

In this chapter, we will first discuss implications and related results of the Calabi-Yau Theorem. We will then go on to reformulate the theorem as an existence and uniqueness theorem of a differential equation of Monge-Ampère-type. Finally, we will give a very rough outline of the steps that are involved in proving it.

4.1 Some implications and related theorems

For $c_1(M) = 0$, we immediately obtain the following result:

4.2 Corollary. Any compact complex manifold with vanishing first Chern class and Kähler structure ω has a unique Ricci-flat Kähler structure ω' cohomologous to ω.

This of course guarantees that every compact Kähler manifold with vanishing first Chern class admits a Calabi-Yau structure.

We call the first Chern class positive or negative if it can be represented by a real $(1,1)$-form that is positive or negative in the sense of 3.29. This yields another straightforward consequence of the Calabi-Yau theorem:

4.3 Corollary. Every compact complex manifold with positive (negative) first Chern class admits a Kähler structure with positive (negative) Ricci form.

Note that we do not have to additionally assume the existence of a Kähler structure here. This is because the real $(1,1)$-form that represents the first Chern class can already supplement the existing complex structure to serve as the Kähler form according to 3.55 and 3.66 (and multiplying with -1 in the negative case).

Since Ricci-flatness just means that a metric is Einstein with proportionality constant zero, Corollary 4.2 can be seen as asserting the existence of a Kähler-Einstein structure with vanishing scalar curvature. It is natural to ask whether one can find a stronger version of Corollary 4.3 that yields existence of Kähler-Einstein metrics with a non-zero proportionality constant. In the case of negative scalar curvature, the following theorem proven independently by both Aubin and Yau answers this question:

4.4 Theorem (Aubin-Calabi-Yau)**.** Every compact complex manifold with negative first Chern class admits a Kähler-Einstein structure with negative scalar curvature.

For a proof, see e.g. chapter 11.C of [19], where the proof of the Calabi-Yau Theorem is extended to cover both at the same time.

4.5 (Existence of Kähler-Einstein metrics with positive scalar curvature)**.** The positive case turns out to be false in general: Remark 11.13 of [19] discusses a class of complex manifolds with positive Chern class, originally constructed in [25], that can be proven to not admit a Kähler-Einstein structure.

However, Tian proposed *K-stability* in [26] as a necessary and sufficient condition on a complex manifold that allows this statement to hold, following a line of reasoning initiated by Yau. In 2012, Chen, Donaldson and Sun [27, 28, 29] proved that this is indeed the case.

4.6 (Some further implications of the Calabi-Yau Theorem)**.** Yau originally attempted to disprove Calabi's conjecture and even announced a counterexample in a 1973 lecture. As part of his attempts, he derived various implications that turned into corollaries once he showed the conjecture to be true instead [30]. Among these are the Bogomolov–Miyaoka–Yau inequality and various existence theorems, announced together with his proof in [23].

4.2 Reformulation as a Monge-Ampère-type equation

Following approximately the discussion in chapter 5.1 of [4], we want to reformulate the Calabi Conjecture as an existence and uniqueness theorem of solutions of a differential equation.

4.7 (Translating forms into functions)**.** The Calabi Conjecture basically asserts that there is a unique way of adjusting the symplectic form ω within its cohomology class to ω' so that the Ricci form ρ changes to any given ρ' in its class. The global dd^c-Lemma 3.79 allows us to express both of these adjustments in terms of real functions $f, \phi \in C^\infty(M)$ that are unique up to a constant:

$$\rho' - \rho = -\frac{1}{2} dd^c f \qquad \text{and} \qquad \omega' - \omega = dd^c \phi, \tag{115}$$

where the factor $-1/2$ serves only to simplify the expressions later on. Moreover, we will fix the constants by also requiring

$$\frac{1}{\mathrm{vol}_\omega(M)} \int_M e^f \mu_\omega = 1 \qquad \text{and} \qquad \int_M \phi \, \mu_\omega = 0, \tag{116}$$

where μ_ω is the volume form associated with ω.

This approach leads to the following equivalent formulation:

4.8 Theorem (Calabi reformulated)**.** Let M be a compact and connected n-manifold equipped with a Kähler structure (J, g, ω). Given any $f \in C^\infty(M)$ so that e^f has average value one, there exists a unique $\phi \in C^\infty(M)$ such that it has both average value zero and

$$(\omega + dd^c\phi)^{n/2} = e^f \omega^{n/2} \tag{117}$$

holds.

4.9 (Local description)**.** Describing the metric induced by $\omega + dd^c\phi$ in local holomorphic coordinates $\{z_a\}_{a=1\ldots n/2}$ yields

$$g'_{a\bar{b}} = g_{a\bar{b}} + \frac{\partial^2 \phi}{\partial z_a \partial \bar{z}_b}. \tag{118}$$

Moreover, since $\omega^{n/2}$ and $(\omega')^{n/2}$ are proportional to their respective volume forms, we can use this and Eq. (85) to rewrite the central differential equation (117) to

$$\det\left(g_{a\bar{b}} + \frac{\partial^2\phi}{\partial z_a \partial \bar{z}_b}\right) = e^f \det\left(g_{a\bar{b}}\right) \tag{119}$$

or equivalently

$$\log\det\left(g_{a\bar{b}} + \frac{\partial^2\phi}{\partial z_a \partial \bar{z}_b}\right) - \log\det\left(g_{a\bar{b}}\right) = f. \tag{120}$$

This is a nonlinear elliptic second-order partial differential equation in ϕ. Since the highest-order terms are given by the determinant of the Hessian matrix of ϕ, it is referred to as being of *Monge-Ampère* type, see e.g. p. 441 of [31].

We will prove the equivalence of the Calabi-Yau theorem and the reformulation 4.8 in four steps:

4.10 Proposition. Let M be a compact and connected manifold of dimension n equipped with a Kähler structure (J, g, ω).

(a) Given any $\phi, f \in C^\infty(M)$ for which Eqs. (116) and (117) hold, $\omega' := \omega + dd^c\phi$ is a Kähler structure (together with the original complex structure J).

(b) Given an additional Kähler structure (J, g', ω'), we find the following equivalence for all $f \in C^\infty(M)$ with $\int_M e^f \mu_g = \mathrm{vol}_g(M)$:

$$(\omega')^{n/2} = e^f \omega^{n/2} \qquad \Longleftrightarrow \qquad \rho' - \rho = -\frac{1}{2} dd^c f \tag{121}$$

(c) The Calabi-Yau Theorem implies 4.8.

(d) 4.8 implies the Calabi-Yau Theorem.

Proof.

(a): We only need to show that ω' and J assemble into a Hermitian structure, since J remains integrable and adding only $dd^c\phi$ clearly keeps the symplectic form closed. Proposition 3.55 then tells us that we are done if ω' is J-invariant and positive definite.

J-invariance follows since d and d^c are real operators, so that $dd^c\phi$ is a real form and necessarily of type $(1,1)$. ω' is positive definite if and only if, in local holomorphic coordinates, the Hermitian matrix $g'_{a\bar{b}}$ from Eq. (118) has only positive eigenvalues. Eq. (119) and positive-definiteness of the original ω already imply that the eigenvalues vanish nowhere, and since they vary continuously on the connected manifold we need to find only one point where they are all positive. As a continuous function on a compact manifold, ϕ has a minimum at a point $p \in M$. At this point, $\partial^2\phi/\partial z_a \partial \bar{z}_b$ has non-negative eigenvalues so that those of $g'_{a\bar{b}}$ are indeed positive.

(b): In local coordinates, the left-hand side of the claimed equivalence takes the form

$$\det\left(g'_{a\bar{b}}\right) = e^f \det\left(g_{a\bar{b}}\right) \tag{122}$$

as a consequence of Eqs. (70) and (85).

By the uniqueness of the global dd^c-Lemma and the representation (98) of ρ' and ρ, we find that the right-hand side $\rho' - \rho = -\frac{1}{2}dd^c f$ is equivalent to

$$\exists c \in \mathbb{R}: \qquad f = \log\frac{\det\left(g'_{a\bar{b}}\right)}{\det\left(g_{a\bar{b}}\right)} + c. \tag{123}$$

Comparing this with Eq. (122), what remains to be shown is that $\int_M e^f \mu_g = \mathrm{vol}_g(M)$ indeed fixes $c = 0$. To this end, we apply the exponential function to Eq. (123) and integrate with respect to the volume form of ω. This yields

$$\int_M e^f \mu_\omega = e^c \int_M \frac{\det\left(g'_{a\bar{b}}\right)}{\det\left(g_{a\bar{b}}\right)} \mu_\omega = e^c \int_M \mu_{\omega'} = e^c \ \mathrm{vol}_{\omega'}(M). \tag{124}$$

Since $[\omega'] = [\omega]$, we find $\int_M \omega^{n/2} = \int_M (\omega')^{n/2}$ and thereby $\mathrm{vol}_{\omega'}(M) = \mathrm{vol}_\omega(M)$ so that the last equation allows us to conclude $c = 0$.

(c): Given the setting of Theorem 4.8, we define the real $(1,1)$-form $\rho' := \rho - \frac{1}{2} dd^c f$ and apply the Calabi-Yau Theorem to it, giving us a Kähler structure expressed through ω' and g'. The global dd^c-Lemma allows us to find a $\phi \in C^\infty(M)$ such that $\omega' - \omega = dd^c\phi$ and $\int_M \phi \mu_g = 0$. (b) then implies that ϕ is indeed a solution to Eq. (117). If there was another such ϕ, (a) and (b) together would imply that this would result in a Kähler structure with the same Ricci form, in contradiction to the uniqueness in the Calabi theorem.

(d): Given the setting of the Calabi-Yau Theorem, we define $f \in C^\infty(M)$ through the global dd^c-Lemma so that $\rho' - \rho = -\frac{1}{2} dd^c f$ and fix the constant as in Eq. (116). 4.8 then hands us a real function ϕ with vanishing average value. Per (a), this defines a new Kähler structure that, by (b), has ρ' as Ricci form. If there was another such Kähler structure, the global dd^c Lemma would yield a $\tilde{\phi}$ with vanishing average that is another solution due to (b), which would violate the uniqueness in Theorem 4.8. □

4.3 Hölder-continuous functions

To discuss the proof of existence, we need to recall some basic facts about Hölder-continuous functions on manifolds due to their convenient regularity results. For more on analysis on manifolds in general, we refer to Aubin's book [32], which also covers this proof.

4.11 (Hölder spaces)**.** Suppose $f \in C^k(M)$ on a Riemannian manifold (M, g), i.e. f has bounded derivatives up to order $k \geq 0$. We call f **C^k-Hölder-continuous with exponent** $\alpha \in (0,1)$ and write $f \in C^{k,\alpha}(M)$ if there exists a $C > 0$ such that, for all $x, y \in M$ whose Riemannian distance $d(x,y)$ is smaller than the injectivity radius $\delta(g)$ of the metric,

$$||\nabla^k f(x) - \nabla^k f(y)|| \leq C\, d(x,y)^\alpha \tag{125}$$

holds. For $k > 0$, the norm on the left hand side has to be understood as the usual norm induced by the metric on the respective tensor space *after* parallely transporting one of the terms along the unique distance-minimizing geodesic that connects x and y.

The norm

$$\begin{aligned} ||f||_{C^{k,\alpha}} :=& \quad ||f||_{C^k(M)} &&+ \quad [f]_\alpha && (126)\\ :=& \sum_{j=0}^{k} \sup_{x\in M} ||\nabla^j f(x)|| &&+ \sup_{\substack{x\neq y\in M\\ d(x,y)<\delta(g)}} \frac{||\nabla^k f(x) - \nabla^k f(y)||}{d(x,y)^a}, && (127)\end{aligned}$$

is well-defined and turns $C^{k,\alpha}(M)$ into a Banach space.

We will need one compact embedding theorem:

4.12 Theorem. Let (M, g) be a compact Riemannian manifold, $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$. The natural embedding $C^{k,\alpha}(M) \hookrightarrow C^k(M)$ is compact, i.e. every bounded subset of $C^{k,\alpha}(M)$ is relatively compact in $C^k(M)$.

See e.g. 2.34 of [32] for a proof.

4.4 Proof outline

This section will sketch an outline of Calabi's and Yau's proof. Again, we follow Joyce [4]. The *continuity method* is used to show existence: We know that the simpler differential equation

$$(\omega + dd^c\phi)^{n/2} = \omega^{n/2}, \tag{128}$$

i.e. if f were zero, is trivially solved by $\phi = 0$. We connect this equation continuously with the actual equation we want to solve by considering those equations corresponding to functions tf for $t \in [0,1]$. To this end, we will use the following definitions:

4.13 Definition. Let M be a compact and connected manifold M with Kähler structure (J, g, ω). We say $f \in C^{3,\alpha}, \phi \in C^{5,\alpha}(M)$ and $A > 0$ **satisfy the Calabi equations** if

$$\int\limits_M \phi \, d\mu_\omega = 0 \quad \text{and} \quad (\omega + dd^c\phi)^{n/2} = Ae^f\omega^{n/2}. \tag{129}$$

Given $\alpha \in (0,1)$ and $f \in C^{3,\alpha}(M)$ we define

$$S := \Big\{ t \in [0,1] \;\Big|\; \exists \phi \in C^{5,\alpha}(M), A > 0 \text{ that satisfy the Calabi equations for } tf \Big\}. \tag{130}$$

While we up to now fixed f in such a way that $A = 1$ always works, we have left A as a variable here instead. By the trivial solution $\phi = 0$ for $t = 0$, we know that S is non-empty. We will rely on the following three auxiliary theorems taken from [4] in order to show that S is both open and closed:

4.14 Theorem C1. Let M be a compact and connected manifold with Kähler structure (J, g, ω) and $Q_1 \geq 0$. There then exist $Q_2, Q_3, Q_4 \geq 0$ such that, for all $f \in C^3(M), \phi \in C^5(M)$ and $A \geq 0$ that satisfy the Calabi equations and

$$||f||_{C^3} \leq Q_1, \tag{131}$$

the following holds:

$$||\phi||_{C^0} \leq Q_2, \qquad ||dd^c\phi||_{C^0} \leq Q_3 \qquad \text{and} \qquad ||\nabla dd^c\phi||_{C^0} \leq Q_4. \tag{132}$$

4.15 Theorem C2. Let M be a compact and connected manifold with Kähler structure (J, g, ω). Let $Q_1, Q_2, Q_3, Q_4 \geq 0$ and $\alpha \in (0,1)$. There then exists a $Q_5 \geq 0$ such that the following holds: For all $f \in C^{3,\alpha}(M), \phi \in C^5(M)$ and $A \geq 0$ that satisfy the Calabi equations and

$$||f||_{C^{3,\alpha}} \leq Q_1, \quad ||\phi||_{C^0} \leq Q_2, \quad ||dd^c\phi||_{C^0} \leq Q_3, \quad \text{and} \quad ||\nabla dd^c\phi||_{C^0} \leq Q_4, \tag{133}$$

it follows that $\phi \in C^{5,\alpha}(M)$ with $||\phi||_{C^{5,\alpha}} \leq Q_5$. If even $f \in C^{k,\alpha}(M)$ for $k \geq$ or $f \in C^\infty(M)$, then $\phi \in C^{k+2,\alpha}(M)$ or $\phi \in C^\infty(M)$, respectively.

4.16 Theorem C3. Let M be a compact and connected manifold with Kähler structure (J, g, ω). For $\alpha \in (0,1), f' \in C^{3,\alpha}(M), \phi' \in C^{5,\alpha}(M), A' > 0$ that satisfy the Calabi equations, it follows that for all $f \in C^{3,\alpha}(M)$ with sufficiently small $||f - f'||_{C^{3,\alpha}}$ there exist $\phi \in C^{5,\alpha}$ and $A > 0$ such that they satisfy the Calabi equations as well.

Showing that S is closed depends on the highly non-trivial a priori estimates C1, which require a series of hard estimates that were supplied by Yau, as well as the regularity results of C2.

Proof that S is a closed subset of $[0,1]$. Suppose t_j is a sequence in S converging to a $t' \in [0,1]$. By definition of S, there exist $\phi_j \in C^{5,\alpha}(M)$ and $A_j > 0$ that satisfy the Calabi equations. Set $Q_1 := ||f||_{C^{3,\alpha}}$ and let Q_2 to Q_5 be the constants resulting from applying first Theorem C1 and then Theorem C2.

Since $t_j \in [0,1]$, we can apply Theorem C1 to ϕ_j, $t_j f$ and A_j without changing the constants. Theorem C2 subsequently guarantees $\phi_j \in C^{5,\alpha}(M)$ with $||\phi_j||_{C^{5,\alpha}} \leq Q_5$. By 4.12, embedding this bounded subset into $C^5(M)$ yields a relatively compact set, which means that a convergent subsequence with limit $\phi' \in C^5(M)$ exists. The A_j of this subsequence converge to $A' := \mathrm{vol}_\omega(M) / \int_M e^{t'f} \mu_\omega > 0$, because one can check analogously to the reasoning around Eq. (124) that the Calabi equations imply that the A_j must be given by $\mathrm{vol}_\omega(M) / \int_M e^{t_j f} \mu_\omega$ and $t_j \to t'$.

Since the ϕ_j converge in $C^5(M)$ and thereby in $C^2(M)$, we can take the limit to see that ϕ' and A' satisfy the Calabi equations for $t_j f$. We have therefore shown that the limit point t' of any sequence $\{t_j\} \subseteq S$ also lies in S, so that S must be closed. ☐

Openness essentially follows from C3, which was already proven by Calabi using the implicit function theorem for Banach spaces.

Proof that S is an open subset on $[0,1]$. Let $t' \in S$, so that there are $\phi' \in C^{5,\alpha}, A' > 0$ that satisfy the Calabi equations. Given a $t \in [0,1]$, use Theorem C3 with $t'f$ and tf instead of f' and f, respectively: $||tf - t'f||_{C^{3,\alpha}} = |t - t'| \cdot ||f||_{C^{3,\alpha}}$ can simultaneously be made small for any t in a small open neighbourhood of t' so that the theorem yields the openness of S. ☐

Now we have assembled everything we need to prove existence:

Proof of Existence in 4.8. S is a non-empty and both open and closed subset of $[0,1]$, so that connectedness of $[0,1]$ implies $S = [0,1]$ and in particular $1 \in S$: For any $f \in C^{3,\alpha}(M)$ we can find a $\phi \in C^{5,\alpha}(M)$ and $A > 0$ that satisfy the Calabi equations. However, in the setting of the reformulated Calabi Theorem 4.8, we even have $f \in C^\infty(M)$ so that also $\phi \in C^\infty(M)$ by Theorem C2. As before, the Calabi equations imply that $A = \mathrm{vol}_\omega(M) / \int_M e^f \mu_\omega$, which just equals 1 since f was chosen in such a way that e^f has average value one. ☐

Uniqueness was already proven by Calabi in [22] and more modern treatments can be found in [4] and [18]. We give a short sketch:

4.17 Theorem C4. Let M be a connected compact complex n-manifold equipped with a Kähler structure ω. For any $f \in C^1(M)$ such that e^f has average value one, there exists at most one $\phi \in C^3(M)$ with average value zero that solves $(\omega + dd^c\phi)^{n/2} = e^f \omega^{n/2}$.

Proof sketch. Let $\omega_{1/2} = \omega + dd^c\phi_{1/2}$ be two solutions to the Calabi equations so that

$$0 = \omega_1^{n/2} - \omega_2^{n/2} = dd^c(\phi_1 - \phi_2) \wedge \sum_{k=0}^{n/2-1} \omega_1^k \wedge \omega_2^{n/2-k-1}. \tag{134}$$

Stokes' Theorem and the fact that $d\omega_{1/2} = 0$ yields

$$0 = \int_M d(\phi_1 - \phi_2) \wedge d^c(\phi_1 - \phi_2) \wedge \sum_{k=0}^{n/2-1} \omega_1^k \wedge \omega_2^{n/2-k-1}. \tag{135}$$

Essentially by using positivity and J-invariance of $\omega_{1/2}$ in local coordinates (see 5.3.5 of [4]), one can see that the $k = 0$ term is proportional to $|d(\phi_1 - \phi_2)|^2$ while the others are non-negative. This means $d(\phi_1 - \phi_2) = 0$ and even $\phi_1 = \phi_2$ since M is connected and $\int_M \phi_{1/2}\, \mu_\omega = 0$. ☐

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